

Kinematics and Statistics of Empirical Currents in Continuous Space at All Times

Cai Dieball and Aljaž Godec*

Mathematical bioPhysics Group, Max Planck Institute for Biophysical Chemistry, Am Fassberg 11, 37077 Göttingen

We present general results on fluctuations and spatial correlations of the empirical density and current of Markovian diffusion in an equilibrium or non-equilibrium steady state on all time scales. We unravel a deep connection between the kinematics of empirical current and “dual-reversal” symmetry. We highlight the essential rôle of coarse-graining in space – the fluctuations of empirical density and current defined in a point are insensitive to non-conservative forces, and are proven to diverge on all time scales in dimensions higher than one, implying non-Gaussian statistics. A spatial coarse-graining is required to uncover salient features of currents due to non-conservative forces. We apply the results to minimal two-dimensional examples of irreversible diffusion. Our findings provide a deeper understanding of time-averaged observables and may allow for a more reliable analysis of single-molecule and particle-tracking experiments.

A non-vanishing probability current [1–17] and entropy production [18–27] are hallmarks of non-equilibrium, manifested as transients during relaxation [25–31] or in non-equilibrium steady states [4–6, 32–34]. Genuinely irreversible, detailed balance violating dynamics emerge in the presence of non-conservative forces (e.g. shear or rotational flow) [35–38] or active driving in living matter fueled by ATP-hydrolysis [16, 39–46]. Such systems are typically small and “soft”, and thus subject to large thermal fluctuations. Single-molecule experiments [45–49] probe non-equilibrium processes on the level of individual finite trajectories that are typically analyzed within the framework of “time-average statistical mechanics” [50, 51] – by averaging along individual realizations yielding random quantities with nontrivial statistics.

Time-average statistical mechanics focuses on functionals of a trajectory $(\mathbf{x}_\tau)_{0 \leq \tau \leq t}$, in particular the empirical density (or occupation time [52–60]) $\rho(\mathbf{x}, t)$ and current $\mathbf{J}(\mathbf{x}, t)$ at a point \mathbf{x} . In laboratory [16, 45] or computer [17] experiments with a finite spatial resolution the empirical density and current are necessarily defined as a spatial average over a window $U_{\mathbf{x}}(\mathbf{x}')$ centered at \mathbf{x} ,

$$\begin{aligned}\overline{\rho^U}(\mathbf{x}, t) &= \frac{1}{t} \int_0^t U_{\mathbf{x}}(\mathbf{x}_\tau) d\tau \\ \overline{\mathbf{J}^U}(\mathbf{x}, t) &= \frac{1}{t} \int_{\tau=0}^{\tau=t} U_{\mathbf{x}}(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau,\end{aligned}\quad (1)$$

where $\circ d\mathbf{x}_\tau$ denotes the Stratonovich integral. These rigorously defined observables are illustrated in Fig. 1. Moreover, $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ (see e.g. [4, 7, 61–68]) correspond to $U_{\mathbf{x}}(\mathbf{z}) = \delta(\mathbf{x} - \mathbf{z})$ where $\delta(\mathbf{z})$ is Dirac’s delta function [69]. If we consider normalized windows $U_{\mathbf{x}}$ [70] then $\overline{\rho^U}(\mathbf{x}, t)$ and $\overline{\mathbf{J}^U}(\mathbf{x}, t)$ are estimators of the probability density and current density, respectively. Conversely, for non-normalized windows [70] $\overline{\rho^U}(\mathbf{x}, t)$ and $\overline{\mathbf{J}^U}(\mathbf{x}, t)$ correspond to estimators of the respective probabilities.

Reliably inferring from experiments whether a noisy system obeys detailed balance, notwithstanding recent

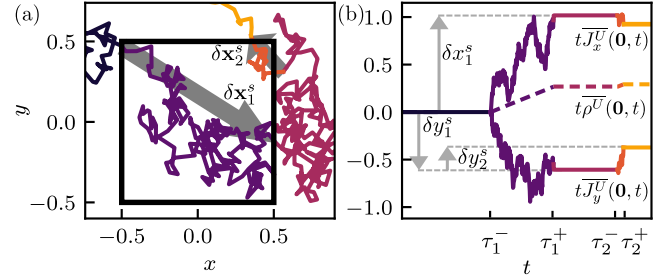


FIG. 1. (a) Schematic of a trajectory traversing an observation window U_0 (black square) with time running from dark to bright. Arrows denote contributions, $\delta \mathbf{x}_i^s = (\delta x_i^s, \delta y_i^s)$, i.e. individual sojourns in U_0 beginning at time τ_i^- and ending at τ_i^+ (see Eqs. (4)); (b) Corresponding $t \rho^U(0, t)$ and components of $t \mathbf{J}^U(0, t)$ from Eq. (1) as functions of t .

progress [16–18, 44–46, 71–73], remains challenging. *Quantifying* violations of detailed balance is a particularly daunting task. One can quantify broken detailed balance through violations of the fluctuation dissipation theorem [33, 74, 75], which requires perturbing the system from the steady state. Alternatively one can check for a symmetry breaking of forward/backward transition-path times [71, 72], or directly measure the entropy production [18, 20, 21, 73] which both require substantial statistics. Most straightforward seems to be the inference of steady-state probability currents [16, 45].

However, single-molecule experiments often cannot reach ergodic times and only allow for a limited number of repetitions. This leads to large uncertainties in the estimates of observables such as steady-state currents. Moreover, current fluctuations have a noise floor – they are bounded from below by the “thermodynamic uncertainty relation” [31, 34, 76, 77]. It is thus imperative to understand and control fluctuations of time-averaged observables [51].

Fluctuating currents in discrete-state jump dynamics are well understood even on an intuitive level [1–3, 5–8, 11–14, 78–84]. Less is known about the statistics of currents in continuous space. There, with important exceptions [4, 9, 15, 85], our understanding is mostly lim-

* agodec@mpibpc.mpg.de

ited to hydrodynamic scales [1, 2, 10] and/or large deviations [7, 61–68]. A comprehensive understanding of fluctuations and spatial correlations of the empirical density and current in continuous space at finite times remains elusive. Moreover, the existing interpretation of large deviations of the empirical current and density defined in a point in physical dimensions higher than one apparently requires a revision (see below).

Here, we address fluctuations and spatial correlations of the empirical density and current in overdamped diffusive steady-state systems and reveal a striking rôle of coarse graining in space. When defined in a point the fluctuations are proven to diverge in physical dimensions larger than one, thus implying non-Gaussian statistics. We explain why a systematic variation of the observation window provides deeper insight about the underlying dynamics. Exploiting a *dual-reversal symmetry* we provide intuition and a deeper understanding of fluctuating currents along individual trajectories. We highlight characteristic differences between reversible and irreversible systems at finite times, which may be used to reliably quantify violations of detailed balance from short measurements. Our results may allow for a more efficient analysis and interpretation of finite, sub-ergodic single-molecule and particle-tracking experiments.

Fundamentals.— We consider time-homogeneous overdamped Langevin dynamics in d -dimensional space [86, 87] described by the Itô stochastic differential equation $d\mathbf{x}_\tau = \mathbf{F}(\mathbf{x}_\tau)d\tau + \boldsymbol{\sigma}d\mathbf{W}_\tau$, where $d\mathbf{W}_\tau$ is the increment of a d -dimensional Wiener processes (i.e. white noise) with zero mean and covariance $\langle dW_{\tau,i}dW_{\tau',j} \rangle = \delta(\tau - \tau')\delta_{ij}dt$. The Itô equation corresponds to the Fokker-Planck equation for the conditional probability density function, which in divergence form reads $(\partial_t + \nabla_{\mathbf{x}} \cdot \hat{\mathbf{j}}_{\mathbf{x}})G(\mathbf{x}, t|\mathbf{y}) = 0$ with initial condition $G(\mathbf{x}, 0|\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$, where we have introduced the current operator $\hat{\mathbf{j}}_{\mathbf{x}} \equiv \mathbf{F}(\mathbf{x}) - \mathbf{D}\nabla_{\mathbf{x}}$ with the positive definite diffusion matrix $\mathbf{D} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T/2$.

We assume the drift field $\mathbf{F}(\mathbf{x})$ to be sufficiently smooth and confining to assure ergodicity, i.e. the existence of an invariant measure with density $G(\mathbf{x}, t \rightarrow \infty|\mathbf{y}) = P_s(\mathbf{x})$ and, if detailed balance is violated, a steady-state probability current $\mathbf{j}_s(\mathbf{x}) \equiv \hat{\mathbf{j}}_{\mathbf{x}}P_s(\mathbf{x}) \neq \mathbf{0}$ [86, 87]. This allows decomposing the drift into orthogonal conservative and incompressible components, i.e. $\mathbf{F}(\mathbf{x}) = \mathbf{D}\{\nabla_{\mathbf{x}}\ln P_s(\mathbf{x})\} + P_s(\mathbf{x})^{-1}\mathbf{j}_s(\mathbf{x})$ [26, 88], implying that $\hat{\mathbf{j}}_{\mathbf{x}}$ decomposes as

$$\hat{\mathbf{j}}_{\mathbf{x}} = P_s^{-1}(\mathbf{x})\mathbf{j}_s(\mathbf{x}) - P_s(\mathbf{x})\mathbf{D}\nabla_{\mathbf{x}}P_s^{-1}(\mathbf{x}) \equiv \hat{\mathbf{j}}^s(\mathbf{x}) + \hat{\mathbf{j}}^g(\mathbf{x}), \quad (2)$$

and obviously $\hat{\mathbf{j}}^g(\mathbf{x})P_s(\mathbf{x}) = 0$. In the examples we will consider a two-dimensional shear flow $\mathbf{F}_{\text{sh}}(\mathbf{x}) = 2x\hat{\mathbf{y}}$ in a box with periodic boundaries (see Fig. 2a-d), and rotational flow in harmonic confinement, $\mathbf{F}_{\text{rot}}(\mathbf{x}) = -\Theta\mathbf{x}$ with $\Theta_{ii} = r > 0$ and $\Theta_{21} = -\Theta_{12} = \Omega$ (see Figs. 2(e-h) and 3); for simplicity we will set \mathbf{D} to be a unit matrix.

For all translation-invariant window functions $U_{\mathbf{x}}(\mathbf{x}') = U_0(\mathbf{x}' - \mathbf{x})$ the time-accumulated density, $t\bar{\rho}^U(\mathbf{x}, t)$, and current, $t\bar{\mathbf{J}}^U(\mathbf{x}, t)$ (see Eq. (1)), are shown

to obey (see proof in Supplementary Material [89])

$$\partial_t[t\bar{\rho}^U(\mathbf{x}, t)] = -\nabla_{\mathbf{x}} \cdot t\bar{\mathbf{J}}^U(\mathbf{x}, t), \quad (3)$$

which *generalizes the notion of the continuity equation to individual trajectories* with arbitrary initial and end points. Taking $U_{\mathbf{x}}$ to be the indicator function of a region with volume h^d centered at \mathbf{x} [69] is particularly enlightening. Namely, by letting the times of entering and exiting said window be τ_i^- and τ_i^+ , respectively, $t\bar{\rho}^U(\mathbf{x}, t)$ is the sum of sojourn times, $\tau_i^s = \tau_i^+ - \tau_i^-$, and $t\bar{\mathbf{J}}^U(\mathbf{x}, t)$ the sum of vectors $\delta\mathbf{x}_i^s$ between corresponding entrance $\mathbf{x}_{\tau_i^-}$ and exit $\mathbf{x}_{\tau_i^+}$ points, that is,

$$\begin{aligned} t\bar{\rho}^U(\mathbf{x}, t) &= \frac{1}{h^d} \sum_{i \leq N_t} (\tau_i^+ - \tau_i^-) \equiv \frac{1}{h^d} \sum_{i \leq N_t} \tau_i^s \\ t\bar{\mathbf{J}}^U(\mathbf{x}, t) &= \frac{1}{h^d} \sum_{i \leq N_t} (\mathbf{x}_{\tau_i^+} - \mathbf{x}_{\tau_i^-}) \equiv \frac{1}{h^d} \sum_{i \leq N_t} \delta\mathbf{x}_i^s, \end{aligned} \quad (4)$$

N_t being the number of visits of the window [90], and \mathbf{x}_0 or \mathbf{x}_t may lie within $U_{\mathbf{x}}$ which we set $\mathbf{x}_{\tau_1^-} = \mathbf{x}_0$ and/or $\mathbf{x}_{\tau_{N_t}^+} \equiv \mathbf{x}_t$. As a result of correlations between $\mathbf{x}_{\tau_i^-}$ and τ_i^s as well as $\mathbf{x}_{\tau_i^+}$ and $\mathbf{x}_{\tau_{i+1}^-}$, $t\bar{\rho}^U$ and $t\bar{\mathbf{J}}^U$ are in general *not* renewal processes. A realization of \mathbf{x}_τ shown in Fig. 1 provides intuition about Eqs. (4).

We throughout assume that trajectories evolve from the steady state $P_s(\mathbf{x})$ such that single-point averages are time-independent, i.e. $\langle \bar{\rho}^U(\mathbf{x}, t) \rangle_s = \int d\mathbf{z} U_{\mathbf{x}}(\mathbf{z}) P_s(\mathbf{z})$ and $\langle \bar{\mathbf{J}}^U(\mathbf{x}, t) \rangle_s = \int d\mathbf{z} U_{\mathbf{x}}(\mathbf{z}) \mathbf{j}_s(\mathbf{z})$, where $\langle \cdot \rangle_s$ denotes the average over all paths \mathbf{x}_τ propagating from $P_s(\mathbf{x})$ (see [89] and [4, 9, 17, 66]). That is, $\langle \bar{\rho}^U(\mathbf{x}, t) \rangle_s$ is an estimator of the steady-state probability density coarse grained over $U_{\mathbf{x}}$, and $\langle \bar{\mathbf{J}}^U(\mathbf{x}, t) \rangle_s$ of the coarse-grained steady-state current, which vanishes under detailed balance (i.e. when $\mathbf{j}_s(\mathbf{x}) = 0$). The limit to a point $U_{\mathbf{x}}(\mathbf{z}) \rightarrow \delta(\mathbf{x} - \mathbf{z})$ [69] yields $\langle \bar{\rho}(\mathbf{x}, t) \rangle_s = P_s(\mathbf{x})$ and $\langle \bar{\mathbf{J}}(\mathbf{x}, t) \rangle_s = \mathbf{j}_s(\mathbf{x})$, respectively, in agreement with existing literature [4, 7, 9, 64–68]. However, stark differences between finite windows and points appear when we consider fluctuations.

Fluctuations and correlations.— Having assumed ergodic dynamics we have $\lim_{t \rightarrow \infty} \bar{\rho}^U(\mathbf{x}, t) = \langle \bar{\rho}^U(\mathbf{x}) \rangle_s$ and $\lim_{t \rightarrow \infty} \bar{\mathbf{J}}^U(\mathbf{x}, t) = \langle \bar{\mathbf{J}}^U(\mathbf{x}) \rangle_s$ [87, 91]. However, at finite times both display large, non-trivial fluctuations. Whereas the mean current vanishes in equilibrium systems, i.e. $\langle \bar{\mathbf{J}}^U(\mathbf{x}) \rangle_s = 0$ under detailed balance, current fluctuations do not. Thus, it is *a priori* not possible to reliably infer from a limited number of finite, sub-ergodic trajectories whether detailed balance is broken. However, a systematic analysis of fluctuations and spatial correlations uncovers qualitative differences between reversible and irreversible dynamics, and establishes a connection with time reversal.

We define the covariance between time-averaged observables evolving from the steady state as

$$C_{AB}^{\mathbf{xy}}(t) \equiv \langle A(\mathbf{x}, t)B(\mathbf{y}, t) \rangle_s - \langle A(\mathbf{x}) \rangle_s \langle B(\mathbf{y}) \rangle_s, \quad (5)$$

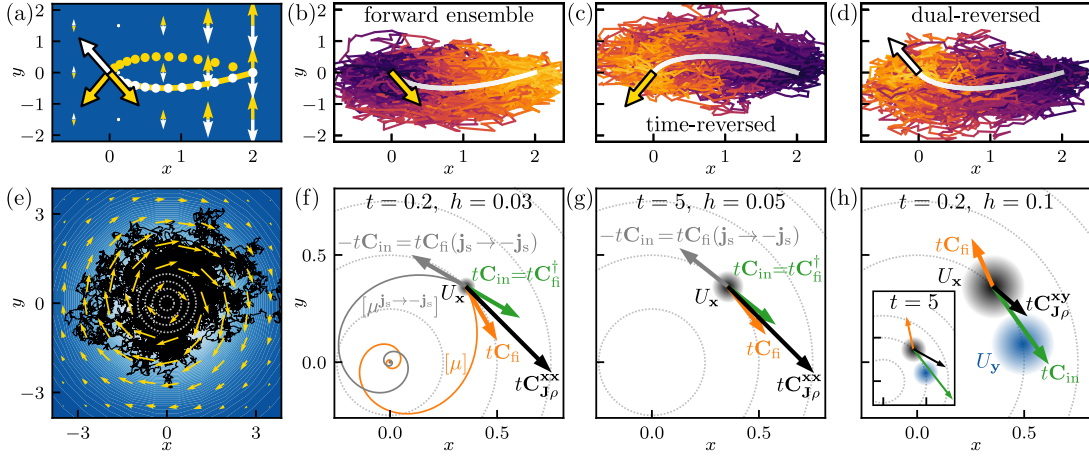


FIG. 2. (a) Shear flow $\mathbf{F}_s(\mathbf{x}')$ (yellow arrows) and its reflection $-\mathbf{F}_s(\mathbf{x}')$ (white arrows); the full line is the average path between two points (see b), yellow and white symbols are average paths in the time-reversed (see c) and dual-reversed (see d) ensemble, respectively; thick arrows show the corresponding currents. (b) Forward, (c) time-reversed, and (d) dual-reversed path ensemble (time runs from dark to bright); the thick line in (b-d) is the average path, arrows depict the (b) initial- and (c-d) final-point current. Dual-reversal symmetry renders the mean paths in (b) and (d) equal (see [89]). (e) Trajectory (black line) in rotational flow, $\mathbf{F}_{\text{rot}}(\mathbf{x}')$, with $r = 1, \Omega = 3$; arrows depict $\mathbf{j}_s(\mathbf{x}')$. (f-g) Single-point $\mathbf{x} = \mathbf{y}$ and (h) two-point time-accumulated correlation $t\mathbf{C}_{J\rho}^{\text{xy}}$ at $t = 0.2$ and $t = 5$ (inset) (black arrow), with final-point $\mathbf{C}_{\text{fi}} \equiv \hat{\mathcal{I}}_{\text{xy}}^t[\hat{\mathbf{j}}_z P_z(\mathbf{z}, t') - \mathbf{j}_s(\mathbf{z}) P_s(\mathbf{z}')]$ (orange) and initial-point $\mathbf{C}_{\text{in}} = \mathbf{C}_{\text{fi}}^\dagger$ (green) contribution, $\mathbf{C}_{J\rho}^{\text{xy}} = \mathbf{C}_{\text{in}} + \mathbf{C}_{\text{fi}}$; $\mathbf{C}_{\text{fi}}^\dagger$ (defined in the text) and $\mathbf{C}_{\text{fi}}(\mathbf{j}_s \rightarrow -\mathbf{j}_s)$ (gray) are the dual- and current-reversed final-point contribution, respectively. Full lines are the mean trajectory $[\mu] \equiv \langle \mathbf{x}_{\tau \geq 0} \rangle_{\mathbf{x}_0 = \mathbf{x}}$ (orange) and its current-reverse $[\mu^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}]$ (gray). $U_{\mathbf{x}, \mathbf{y}}$ (shaded circles) is a Gaussian at \mathbf{x}, \mathbf{y} with width h .

where A and B are either $\overline{\rho^U}$ or $\overline{\mathbf{J}^U}$, respectively, and we used the fact that single-point steady-state averages are time-independent. We refer to the case $A \neq B$ or $\mathbf{x} \neq \mathbf{y}$ as (linear) correlations and to $A = B$ with $\mathbf{x} = \mathbf{y}$ as “fluctuations” with the convention $C_{AA}^{\text{xx}}(t) \rightarrow \text{var}_A^{\text{xx}}(t)$. Using Itô calculus we derive in [89] a Feynman-Kac equation for the generating function of the joint density of $\overline{\rho^U}$ and $\overline{\mathbf{J}^U}$ where from we determine $C_{AB}^{\text{xy}}(t)$. We introduce the two-time steady-state joint density $P_{\mathbf{y}}(\mathbf{x}, t) \equiv G(\mathbf{x}, t|\mathbf{y})P_s(\mathbf{y})$ as well as the integral operator

$$\hat{\mathcal{I}}_{\text{xy}}^t[\cdot] = \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t}\right) \int d\mathbf{z} \int d\mathbf{z}' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{y}}(\mathbf{z}') [\cdot], \quad (6)$$

which allow us to write the *exact two-point correlation of the empirical density* in the form (see [89] and [51, 53])

$$C_{\rho\rho}^{\text{xy}}(t) = \hat{\mathcal{I}}_{\text{xy}}^t[P_{\mathbf{z}'}(\mathbf{z}, t') + P_{\mathbf{z}}(\mathbf{z}', t') - 2P_s(\mathbf{z})P_s(\mathbf{z}')]. \quad (7)$$

Under detailed balance $P_{\mathbf{z}}(\mathbf{z}', t) = P_{\mathbf{z}'}(\mathbf{z}, t)$ and Eq. (7) reduces to $C_{\rho\rho}^{\text{xy}}(t) = 2\hat{\mathcal{I}}_{\text{xy}}^t[P_{\mathbf{z}'}(\mathbf{z}, t') - P_s(\mathbf{z})P_s(\mathbf{z}')]$.

To carry out the analysis of $\overline{\mathbf{J}^U}$ we exploit a *dual reversal invariance for systems arbitrarily far from equilibrium*, and relate correlations of $\overline{\rho^U}$ and $\overline{\mathbf{J}^U}$ to initial- and final-point probability currents along steady state paths.

Consider the dynamics between fixed points $\mathbf{x} \rightarrow \mathbf{y}$ in fixed time t (see Fig. 2a-d). We define the *initial-* and *final-point* currents at \mathbf{x} and \mathbf{y} , respectively, as

$$\begin{aligned} \mathbf{j}_{\text{in}}^{\text{y}}(\mathbf{x}, t) dt &\equiv \langle \delta(\mathbf{y} - \mathbf{x}_t) \delta(\mathbf{x} - \mathbf{x}_0) \circ d\mathbf{x}_0 \rangle_s \\ \mathbf{j}_{\text{fi}}^{\text{x}}(\mathbf{y}, t) dt &\equiv \langle \delta(\mathbf{x} - \mathbf{x}_0) \delta(\mathbf{y} - \mathbf{x}_t) \circ d\mathbf{x}_t \rangle_s, \end{aligned} \quad (8)$$

yielding (see [89]) $\mathbf{j}_{\text{fi}}^{\text{x}}(\mathbf{y}, t) = \hat{\mathbf{j}}_{\text{y}} P_{\mathbf{x}}(\mathbf{y}, t)$. The analysis of $\mathbf{j}_{\text{in}}^{\text{y}}(\mathbf{x}, t)$ leads to *dual dynamics* $\mathbf{x}_{t-\tau}^\dagger$ [21, 92–96], i.e. time-reversal with concurrent reflection of $\mathbf{j}_s(\mathbf{x})$. Dual-reversal symmetry implies (see Fig. 2a-d and [89] for the proof)

$$P_{\mathbf{x}}(\mathbf{y}, t) = P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) \implies \mathbf{j}_{\text{in}}^{\text{y}}(\mathbf{x}, t) = \hat{\mathbf{j}}_{\text{x}}^\dagger P_{\mathbf{x}}(\mathbf{y}, t), \quad (9)$$

where $P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) \equiv \langle \delta(\mathbf{y} - \mathbf{x}_t^\dagger) \delta(\mathbf{x} - \mathbf{x}_0^\dagger) \rangle_s^\dagger$ is the two-time steady-state density in the dual ensemble $\langle \cdot \rangle_s^\dagger$, and $\hat{\mathbf{j}}_{\text{x}}^\dagger$ the corresponding dual-reversed current operator $\hat{\mathbf{j}}_{\text{x}}^\dagger \equiv \hat{\mathbf{j}}_s(\mathbf{x}) - \hat{\mathbf{j}}_s^\dagger(\mathbf{x})$. Under detailed balance (i.e. $\mathbf{j}_s(\mathbf{x}) = 0$) $P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) = P_{\mathbf{y}}(\mathbf{x}, t) = P_{\mathbf{x}}(\mathbf{y}, t)$. This allows interpreting our first main result (see proof in [89])

$$\mathbf{C}_{J\rho}^{\text{xy}}(t) = \hat{\mathcal{I}}_{\text{xy}}^t[\hat{\mathbf{j}}_z P_z(\mathbf{z}, t') + \hat{\mathbf{j}}_z^\dagger P_z(\mathbf{z}', t') - 2\mathbf{j}_s(\mathbf{z}) P_s(\mathbf{z}')], \quad (10)$$

as a vector with initial- and final-point contributions, $\mathbf{C}_{J\rho}^{\text{xy}} = \mathbf{C}_{\text{in}} + \mathbf{C}_{\text{fi}}$, where $\mathbf{C}_{\text{in}} \equiv \hat{\mathcal{I}}_{\text{xy}}^t[\hat{\mathbf{j}}_z^\dagger P_z(\mathbf{z}', t') - \mathbf{j}_s(\mathbf{z}) P_s(\mathbf{z}')]$ (see Fig. 2e-h). In detailed balance $\mathbf{C}_{J\rho}^{\text{xy}}(t) = 0$ since $\mathbf{j}_s(\mathbf{x}) = 0$ and thus $\hat{\mathbf{j}}_z^\dagger = -\hat{\mathbf{j}}_z$. A non-zero $\mathbf{C}_{J\rho}^{\text{xy}}(t)$ at any time t (see example in Fig. 2f-h) is thus a *conclusive* signature of broken detailed balance.

For $\mathbf{x} = \mathbf{y}$ and small windows where $\langle \overline{\mathbf{J}^U}(\mathbf{x}) \rangle_s \approx \mathbf{j}_s(\mathbf{x})$, \mathbf{C}_{in} corresponds to the dual-reversed \mathbf{C}_{fi} , i.e. $\mathbf{C}_{\text{in}} = \mathbf{C}_{\text{fi}}^\dagger \equiv \mathbf{C}_{\text{fi}}(\mathbf{z} \leftrightarrow \mathbf{z}', \hat{\mathbf{j}} \rightarrow \hat{\mathbf{j}}^\dagger)$, geometrically implying that $\mathbf{C}_{J\rho}^{\text{xx}}$ points along $\mathbf{j}_s(\mathbf{x})$ (see Fig. 2f-g). For small t or very small windows (see small-window limit below), \mathbf{C}_{fi} points along $\mathbf{F}(\mathbf{x})$ that is tangent to the mean trajectory $[\mu]$ at \mathbf{x} (see Fig. 2f). Conversely, the two-point correlation $\mathbf{C}_{J\rho}^{\text{xy}}$ need

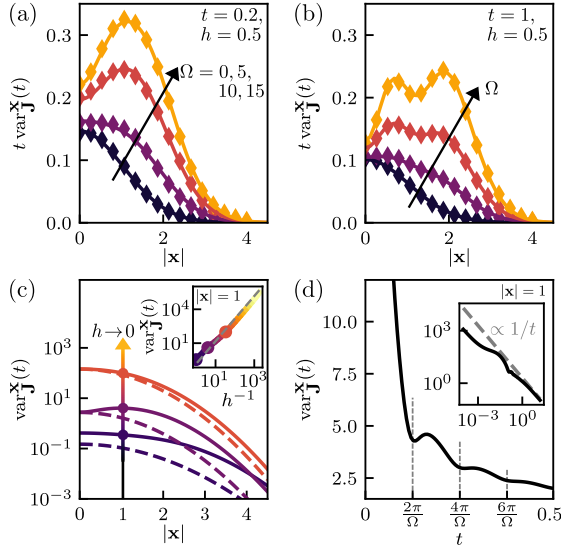


FIG. 3. $t \text{var}_{\mathbf{J}}^{\mathbf{x}}$ as a function of the radius $|\mathbf{x}|$ in $\mathbf{F}_{\text{rot}}(\mathbf{x}')$ (see Fig. 2e) upon increasing Ω for Gaussian $U_{\mathbf{x}}$ with width h at (a) $t = 0.2$ and (b) $t = 1$; lines depict Eq. (11) and symbols Brownian dynamics simulations (see [89]). (c) $t \text{var}_{\mathbf{J}}^{\mathbf{x}}$ (Eq. (11)) at $t = 1$ for $\Omega = 10$ (full lines) and equilibrium, $\Omega = 0$ (dashed lines), for various h decreasing along the arrow. Inset: divergence of $\text{var}_{\mathbf{J}}^{\mathbf{x}}$ as $h \rightarrow 0$ at $|\mathbf{x}| = 1$; the dashed line depicts Eq. (13). Note the logarithmic scales. (d) $\text{var}_{\mathbf{J}}^{\mathbf{x}}$ as a function of t for strong driving $\Omega = 50$; Inset: (d) on logarithmic scales alongside the large deviation scaling $\propto t^{-1}$.

not to point along $\mathbf{j}_s(\mathbf{x})$ (Fig. 2h). In fact, its direction changes over time (see inset of Fig. 2h).

We now address fluctuations and two-point correlations of the empirical current that are characterized by the $d \times d$ covariance matrix with elements $(C_{\mathbf{J}\mathbf{J}}^{\mathbf{xy}}(t))_{ik} = C_{\mathbf{J}_i\mathbf{J}_k}^{\mathbf{xy}}(t)$. We focus on the scalar case $C_{\mathbf{J}\mathbf{J}}^{\mathbf{xy}}(t) \equiv \text{Tr} C_{\mathbf{J}\mathbf{J}}^{\mathbf{xy}}(t)$, which can be written as (see proof in [89])

$$C_{\mathbf{J}\mathbf{J}}^{\mathbf{xy}}(t) = \frac{2 \text{Tr} \mathbf{D}}{t} \int d\mathbf{z} U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{y}}(\mathbf{z}) P_s(\mathbf{z}) + \hat{\mathcal{I}}_{\mathbf{xy}}^t [\hat{\mathbf{j}}_{\mathbf{z}} \cdot \hat{\mathbf{j}}_{\mathbf{z}'}^{\dagger} P_{\mathbf{z}'}(\mathbf{z}, t') + \hat{\mathbf{j}}_{\mathbf{z}'} \cdot \hat{\mathbf{j}}_{\mathbf{z}}^{\dagger} P_{\mathbf{z}}(\mathbf{z}', t') - 2\mathbf{j}_s(\mathbf{z}) \cdot \mathbf{j}_s(\mathbf{z}')]. \quad (11)$$

At equilibrium the second term in Eq. (11) reduces to $-2\hat{\mathcal{I}}_{\mathbf{xy}}^t [\hat{\mathbf{j}}^g(\mathbf{z}) \cdot \hat{\mathbf{j}}^g(\mathbf{z}') P_{\mathbf{z}'}(\mathbf{z}, t')]$, which does not vanish although $\langle \bar{\mathbf{J}}^U(\mathbf{x}) \rangle_s = 0$. Eq. (11) relates the (co)variance of $\bar{\mathbf{J}}^U(\mathbf{x}, t)$ to initial- and final-point currents, and is our second main result. Notably, when $\mathbf{j}_s \neq \mathbf{0}$, $\text{var}_{\mathbf{J}}^{\mathbf{x}}(t) \equiv C_{\mathbf{J}\mathbf{J}}^{\mathbf{xx}}(t)$ may display maxima where $P_s(\mathbf{x})$ has none (see Fig. 3(a-c)), and an oscillatory time-dependence due to circulating currents (see Fig. 3d), both signaling non-equilibrium.

Small windows and large deviations. — Because Markovian diffusion in dimensions $d \geq 2$ hits a point with zero probability we expect qualitative differences between fluctuations of $\bar{\rho}^U(\mathbf{x}, t)$ and $\bar{\mathbf{J}}^U(\mathbf{x}, t)$ defined in a finite window and that defined in a point. Indeed, in the limit of small windows with a width h the variance and covari-

ance of $\bar{\rho}^U$ and $\bar{\mathbf{J}}^U$ behave as

$$\text{var}_{\rho}^{\mathbf{x}}(t) \stackrel{h \rightarrow 0}{\simeq} \frac{K}{\tilde{D}t} P_s(\mathbf{x}) \begin{cases} \frac{h^{2-d}}{d-2} & \text{for } d > 2 \\ -\ln h & \text{for } d = 2 \end{cases}, \quad (12)$$

where \simeq denotes asymptotic equality. Moreover, we have $C_{\mathbf{J}\rho}^{\mathbf{xx}}(t) \simeq \mathbf{j}_s(\mathbf{x}) \text{var}_{\rho}^{\mathbf{x}}(t) / 2P_s(\mathbf{x})$ with contributions $\mathbf{C}_{\text{in}}(t) \simeq [2\mathbf{j}_s(\mathbf{x})/P_s(\mathbf{x}) - \mathbf{F}(\mathbf{x})] \text{var}_{\rho}^{\mathbf{x}}(t) / 4$ and $\mathbf{C}_{\text{fi}}(t) \simeq \mathbf{F}(\mathbf{x}) \text{var}_{\rho}^{\mathbf{x}}(t) / 4$. The variance of $\bar{\mathbf{J}}^U$ obeys

$$\text{var}_{\mathbf{J}}^{\mathbf{x}}(t) \stackrel{h \rightarrow 0}{\simeq} K' \frac{2\tilde{D}'}{t} P_s(\mathbf{x}) (d-1) h^{-d} + \mathcal{O}(t^{-1}) \mathcal{O}(h^{1-d}), \quad (13)$$

with $\tilde{D}, \tilde{D}' \in [\lambda_{\min}, \lambda_{\max}]$ bounded by the eigenvalues of \mathbf{D} , and K, K' are constants depending on the specific normalized window $U_{\mathbf{z}}$ (see [89]). Moreover, taking $U_{\mathbf{x}}(\mathbf{z}) \rightarrow \delta(\mathbf{x} - \mathbf{z})$ as in [7, 51, 61–68] we find for $d \geq 2$ that $\text{var}_{\rho, \mathbf{J}}^{\mathbf{xx}}(t), C_{\mathbf{J}\rho}^{\mathbf{xx}}(t)$ diverge for all t (see example in Fig. 3c). Notably, the two limits $h \rightarrow 0$ and $t \rightarrow \infty$ do not commute. This has important consequences for *large deviations*, i.e. the statistics on ergodic time-scales.

The large deviation principle [61, 66, 91] characterizes fluctuations around a *typical* (and concurrently mean) value of observables whose probability density is assumed to depend exponentially on time, i.e. $\lim_{t \rightarrow \infty} t^{-1} \ln p(A_t = a) = -I(a)$, where A_t is here $\bar{\rho}^U(\mathbf{x}, t)$ or $\bar{\mathbf{J}}^U(\mathbf{x}, t)$, and $I(a) \geq 0$ is the “rate function”. For non-zero h or in $d = 1$, $I(a)$ is a parabola at $\langle A_t \rangle$ with coefficient $1/2\sigma_A^2$, where σ_A^2 is obtained by replacing $\hat{\mathcal{I}}_{\mathbf{xx}}^t$ in Eq. (6) with $\int_0^\infty dt' \int d\mathbf{z} \int d\mathbf{z}' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}')$ (see Fig. S2 in [89]).

The divergence of $\text{var}_{\rho, \mathbf{J}}^{\mathbf{x}}$ defined in a point (i.e. $h = 0$) in dimensions $d \geq 2$ implies non-Gaussian large deviations [97] and, since $\langle A_t \rangle$ is finite, the existence of power-law tails [97] with an exponent $\alpha \leq 3$. Thus, if at long times $p(A_t = a)$ is exponential in t , $I(a) \propto \alpha \ln(a)$ for large a , while $I(a) = 0$ or $I(a) = \infty$ if it is sub- or super-exponential, respectively [61]. In any case $I(a)$ is *not* strictly convex and the Gärtner-Ellis theorem [61, 66] must *not* be applied to determine $I(a)$ for $d \geq 2$ when $U_{\mathbf{x}}(\mathbf{z}) = \delta(\mathbf{x} - \mathbf{z})$. However, using instead a Dirac measure $U_{\mathbf{x}}(\mathbf{z}) = \delta_{\mathbf{x}}(\mathbf{z})$ – the indicator function of a point – yields an estimator for the *probability* in a point (which is a.s. zero), and one may use the Gärtner-Ellis theorem to prove a large deviation principle for *measures* [98].

Notably, for small windows Eqs. (S75-13) imply that fluctuations (unlike correlations) carry no information about steady-state currents $\mathbf{j}_s(\mathbf{x})$ and thus violations of detailed balance. In this limit fluctuations reflect only Brownian, thermal currents (see [89]) that are invariant with respect to $\mathbf{j}_s(\mathbf{x})$ – systems with equal $P_s(\mathbf{x})$ and \mathbf{D} display the same fluctuations (see Fig. 3c). Only intermediate windows $U_{\mathbf{x}}$ reveal spatial features of steady-state currents (see Fig. 3c), which invites further analysis.

Conclusion. — We derived general results on fluctuations and spatial correlations of empirical density and current in continuous-space steady states at all times, which revealed striking effects of spatial coarse-graining.

The fluctuations defined in a point were shown to carry no information about the non-conservative part of the drift, and were found to diverge in dimensions higher than one, which may be important when generalizing results derived for discrete or one-dimensional systems, e.g. the thermodynamic uncertainty relation [5, 31, 34, 77, 83, 99], to multi-dimensional continuous space. A spatial coarse-graining was shown to be required to uncover salient features of currents due to non-conservative forces, without inferring these individually [100–102]. We connected the kinematics of empirical current to the “dual-reversed” dynamics thus providing much desired intuition, which may concurrently deepen our understand-

ing of the asymmetry in relaxation towards equilibrium [27]. Non-vanishing correlations between empirical current and density were shown to be a conclusive indicator of broken detailed balance, and may further improve the accuracy of inferring invariant densities [103]. Our results allow for generalizations to transients due to non-stationary initial conditions or non-ergodic dynamics, which will be addressed in forthcoming publications.

Acknowledgments.— The financial support from Studienstiftung des Deutschen Volkes (to C. D.) and the German Research Foundation (DFG) through the Emmy Noether Program GO 2762/1-1 (to A. G.) is gratefully acknowledged.

-
- [1] T. Bodineau and B. Derrida, Phys. Rev. Lett. **92**, 180601 (2004).
 - [2] L. Bertini, A. D. Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Phys. Rev. Lett. **94**, 030601 (2005).
 - [3] R. K. P. Zia and B. Schmittmann, J. Stat. Mech: Theory Exp. , 07012 (2007).
 - [4] C. Maes, K. Netočný, and B. Wynants, Physica A **387**, 2675 (2008).
 - [5] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Phys. Rev. Lett. **116**, 120601 (2016).
 - [6] C. Maes and K. Netočný, EPL (Europhys. Lett.) **82**, 30003 (2008).
 - [7] A. C. Barato and R. Chetrite, J. Stat. Phys. **160**, 1154 (2015).
 - [8] M. Baiesi, C. Maes, and K. Netočný, J. Stat. Phys. **135**, 57 (2009).
 - [9] V. Y. Chernyak, M. Chertkov, S. V. Malinin, and R. Teodorescu, J. Stat. Phys. **137**, 109 (2009).
 - [10] L. Bertini, A. D. Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Rev. Mod. Phys. **87**, 593 (2015).
 - [11] P. Pietzonka, A. C. Barato, and U. Seifert, Phys. Rev. E **93**, 052145 (2016).
 - [12] T. R. Gingrich and J. M. Horowitz, Phys. Rev. Lett. **119**, 170601 (2017).
 - [13] A. C. Barato, R. Chetrite, A. Faggionato, and D. Gabrielli, New J. Phys. **20**, 103023 (2018).
 - [14] M. Kaiser, R. L. Jack, and J. Zimmer, J. Stat. Phys. **170**, 1019 (2018).
 - [15] A. Dechant and S.-i. Sasa, J. Stat. Mech: Theory Exp. , 063209 (2018).
 - [16] C. Battle, C. P. Broedersz, N. Fakhri, V. F. Geyer, J. Howard, C. F. Schmidt, and F. C. MacKintosh, Science **352**, 604 (2016).
 - [17] J. Li, J. M. Horowitz, T. R. Gingrich, and N. Fakhri, Nat. Commun. **10**, 1666 (2019).
 - [18] É. Roldán and J. M. R. Parrondo, Phys. Rev. Lett. **105**, 150607 (2010).
 - [19] L. Dabelow, S. Bo, and R. Eichhorn, Phys. Rev. X **9**, 021009 (2019).
 - [20] S. Pigolotti, I. Neri, É. Roldán, and F. Jülicher, Phys. Rev. Lett. **119**, 140604 (2017).
 - [21] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
 - [22] U. Seifert, Phys. Rev. Lett. **95**, 040602 (2005).
 - [23] M. Esposito and C. Van den Broeck, Phys. Rev. E **82**, 011143 (2010).
 - [24] C. Van den Broeck and M. Esposito, Phys. Rev. E **82**, 011144 (2010).
 - [25] S. Vaikuntanathan and C. Jarzynski, EPL (Europhys. Lett.) **87**, 60005 (2009).
 - [26] H. Qian, J. Math. Phys. **54**, 053302 (2013).
 - [27] A. Lapolla and A. Godec, Phys. Rev. Lett. **125**, 110602 (2020).
 - [28] C. Maes, K. Netočný, and B. Wynants, Phys. Rev. Lett. **107**, 010601 (2011).
 - [29] C. Maes, Phys. Rev. Lett. **119**, 160601 (2017).
 - [30] N. Shiraishi and K. Saito, Phys. Rev. Lett. **123**, 110603 (2019).
 - [31] T. Koyuk and U. Seifert, Phys. Rev. Lett. **125**, 260604 (2020).
 - [32] D.-Q. Jiang, M. Qian, and M.-P. Qian, *Mathematical Theory of Nonequilibrium Steady States* (Springer Berlin Heidelberg, 2004).
 - [33] U. Seifert and T. Speck, EPL (Europhys. Lett.) **89**, 10007 (2010).
 - [34] A. C. Barato and U. Seifert, Phys. Rev. Lett. **114**, 158101 (2015).
 - [35] C. Schroeder, R. Teixeira, E. Shaqfeh, and S. Chu, Phys. Rev. Lett. **95**, 018301 (2005).
 - [36] M. Harasim, B. Wunderlich, O. Peleg, M. Kröger, and A. R. Bausch, Phys. Rev. Lett. **110**, 108302 (2013).
 - [37] S. Gerashchenko and V. Steinberg, Phys. Rev. Lett. **96**, 038304 (2006).
 - [38] A. Alexander-Katz, M. F. Schneider, S. W. Schneider, A. Wixforth, and R. R. Netz, Phys. Rev. Lett. **97**, 138101 (2006).
 - [39] H. Qian and M. Qian, Phys. Rev. Lett. **84**, 2271 (2000).
 - [40] H. Qian, Annu. Rev. Phys. Chem. **58**, 113 (2007).
 - [41] S. Toyabe, T. Okamoto, T. Watanabe-Nakayama, H. Taketani, S. Kudo, and E. Muneyuki, Phys. Rev. Lett. **104**, 198103 (2010).
 - [42] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, Rev. Mod. Phys. **85**, 1143 (2013).
 - [43] N. Fakhri, A. D. Wessel, C. Willms, M. Pasquali, D. R. Klopfenstein, F. C. MacKintosh, and C. F. Schmidt, Science **344**, 1031 (2014).
 - [44] É. Fodor, C. Nardini, M. E. Cates, J. Tailleur, P. Visco, and F. van Wijland, Phys. Rev. Lett. **117**, 038103 (2016).
 - [45] J. Gladrow, N. Fakhri, F. C. MacKintosh, C. F. Schmidt, and C. P. Broedersz, Phys. Rev. Lett. **116**, 248301 (2016).

- (2016).
- [46] F. S. Gnesotto, F. Mura, J. Gladrow, and C. P. Broedersz, *Rep. Prog. Phys.* **81**, 066601 (2018).
 - [47] F. Ritort, *J. Phys.: Cond. Matt.* **18**, R531 (2006).
 - [48] W. J. Greenleaf, M. T. Woodside, and S. M. Block, *Annu. Rev. Biophys. Biomol. Struct.* **36**, 171 (2007).
 - [49] J. R. Moffitt, Y. R. Chemla, S. B. Smith, and C. Bustamante, *Annu. Rev. Biochem.* **77**, 205 (2008).
 - [50] S. Burov, J.-H. Jeon, R. Metzler, and E. Barkai, *Phys. Chem. Chem. Phys.* **13**, 1800 (2011).
 - [51] A. Lapolla, D. Hartich, and A. Godec, *Phys. Rev. Research* **2**, 043084 (2020).
 - [52] M. Kac, *Trans. Am. Math. Soc.* **65**, 1 (1949).
 - [53] D. A. Darling and M. Kac, *Trans. Am. Math. Soc.* **84**, 444 (1957).
 - [54] E. Aghion, D. A. Kessler, and E. Barkai, *Phys. Rev. Lett.* **122**, 010601 (2019).
 - [55] S. Carmi and E. Barkai, *Phys. Rev. E* **84**, 061104 (2011).
 - [56] S. N. Majumdar and A. Comtet, *Phys. Rev. Lett.* **89**, 060601 (2002).
 - [57] S. N. Majumdar and D. S. Dean, *Phys. Rev. E* **66**, 041102 (2002).
 - [58] S. N. Majumdar, *Curr. Sci.* **89**, 2075 (2005).
 - [59] A. J. Bray, S. N. Majumdar, and G. Schehr, *Adv. Phys.* **62**, 225 (2013).
 - [60] G. Bel and E. Barkai, *Phys. Rev. Lett.* **94**, 240602 (2005).
 - [61] H. Touchette, *Phys. Rep.* **478**, 1 (2009).
 - [62] S. Kusuoka, K. Kuwada, and Y. Tamura, *Probab. Theory Relat. Fields* **147**, 649 (2009).
 - [63] R. Chetrite and H. Touchette, *Phys. Rev. Lett.* **111**, 120601 (2013).
 - [64] R. Chetrite and H. Touchette, *Annales Henri Poincaré* **16**, 2005 (2014).
 - [65] J. Hoppenau, D. Nickelsen, and A. Engel, *New J. Phys.* **18**, 083010 (2016).
 - [66] H. Touchette, *Physica A* **504**, 5 (2018).
 - [67] E. Mallmin, J. du Buisson, and H. Touchette, *Large deviations of currents in diffusions with reflective boundaries* (2021), arXiv:2102.04846 [cond-mat.stat-mech].
 - [68] C. Monthus, *Inference of Markov models from trajectories via Large Deviations at level 2.5 with applications to random walks in disordered media* (2021), arXiv:2101.09045 [cond-mat.stat-mech].
 - [69] We consider L^1 functions e.g. a Gaussian window or a normalized indicator function of a region $\Omega_{\mathbf{x}}$ with volume h^d , $U_{\mathbf{x},h}(\mathbf{z}) = h^{-d}\mathbb{1}_{\Omega_{\mathbf{x}}}(\mathbf{z})$ such that $\lim_{h \rightarrow 0} U_{\mathbf{x},h}(\mathbf{z}) = \delta(\mathbf{x} - \mathbf{z})$.
 - [70] $U_{\mathbf{x}}$ is said to be normalized if $\int U_{\mathbf{x}}(\mathbf{z})d\mathbf{z} = 1$, and non-normalized when $\int U_{\mathbf{x}}(\mathbf{z})d\mathbf{z} = V$, where V is the volume of the window.
 - [71] J. Gladrow, M. Ribezzi-Crivellari, F. Ritort, and U. F. Keyser, *Nat. Commun.* **10**, 10.1038/s41467-018-07873-9 (2019).
 - [72] A. M. Berezhkovskii and D. E. Makarov, *J. Phys. Chem. Lett.* **11**, 1682 (2020).
 - [73] D. S. Seara, B. B. Machta, and M. P. Murrell, *Nat. Commun.* **12**, 10.1038/s41467-020-20281-2 (2021).
 - [74] L. F. Cugliandolo, D. S. Dean, and J. Kurchan, *Phys. Rev. Lett.* **79**, 2168 (1997).
 - [75] D. Mizuno, C. Tardin, C. F. Schmidt, and F. C. MacKintosh, *Science* **315**, 370 (2007).
 - [76] U. Seifert, *Physica A* **504**, 176 (2018).
 - [77] J. M. Horowitz and T. R. Gingrich, *Nat. Phys.* **16**, 15 (2019).
 - [78] J. L. Lebowitz and H. Spohn, *J. Stat. Phys.* **95**, 333 (1999).
 - [79] H. Qian, S. Saffarian, and E. L. Elson, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 10376 (2002).
 - [80] S. Pilgram, A. N. Jordan, E. V. Sukhorukov, and M. Büttiker, *Phys. Rev. Lett.* **90**, 206801 (2003).
 - [81] D. Andrieux and P. Gaspard, *J. Stat. Phys.* **127**, 107 (2007).
 - [82] S. C. Kapfer and W. Krauth, *Phys. Rev. Lett.* **119**, 240603 (2017).
 - [83] K. Macieszczak, K. Brandner, and J. P. Garrahan, *Phys. Rev. Lett.* **121**, 130601 (2018).
 - [84] S. Marcantoni, C. Pérez-Espigares, and J. P. Garrahan, *Phys. Rev. E* **101**, 062142 (2020).
 - [85] F. Flandoli, M. Gubinelli, M. Giaquinta, and V. M. Tortorelli, *Stochastic Processes Appl.* **115**, 1583 (2005).
 - [86] C. W. Gardiner, *Handbook of stochastic methods for physics, chemistry, and the natural sciences* (Springer-Verlag, Berlin New York, 1985).
 - [87] G. A. Pavliotis, *Stochastic Processes and Applications* (Springer New York, 2014).
 - [88] A. Lapolla and A. Godec, *Front. Phys.* **7**, 182 (2019).
 - [89] See Supplemental Material at [...] for detailed calculations and mathematical proofs and auxiliary results.
 - [90] Note that N_t is almost surely either ∞ or 0.
 - [91] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications* (Springer Berlin Heidelberg, 2010).
 - [92] T. Hatano and S.-i. Sasa, *Phys. Rev. Lett.* **86**, 3463 (2001).
 - [93] R. Chetrite and K. Gawędzki, *Commun. Math. Phys.* **282**, 469 (2008).
 - [94] C. Maes and K. Netočný, *J. Stat. Phys.* **154**, 188 (2013).
 - [95] M. Esposito and C. Van den Broeck, *Phys. Rev. Lett.* **104**, 090601 (2010).
 - [96] A. Dechant and S. ichi Sasa, *Continuous time-reversal and equality in the thermodynamic uncertainty relation* (2021), arXiv:2010.14769 [cond-mat.stat-mech].
 - [97] W. Feller, *An introduction to probability theory and its applications. Vol. I and II*, Third edition (John Wiley & Sons Inc., New York, 1968).
 - [98] G. Ferré and G. Stoltz, *Electron. J. Probab.* **25**, 1 (2020).
 - [99] K. Liu, Z. Gong, and M. Ueda, *Phys. Rev. Lett.* **125**, 140602 (2020).
 - [100] A. Frishman and P. Ronceray, *Phys. Rev. X* **10**, 021009 (2020).
 - [101] D. B. Brückner, P. Ronceray, and C. P. Broedersz, *Phys. Rev. Lett.* **125**, 058103 (2020).
 - [102] F. Ferretti, V. Chardès, T. Mora, A. M. Walczak, and I. Giardina, *Phys. Rev. X* **10**, 031018 (2020).
 - [103] D. Hartich and U. Seifert, *Phys. Rev. E* **94**, 042416 (2016).

Supplementary Material for: Kinematics and Statistics of Empirical Currents in Continuous Space at all Times

Cai Dieball and Aljaž Godec

Mathematical bioPhysics Group, Max Planck Institute for Biophysical Chemistry, 37077 Göttingen, Germany

In this Supplementary Material (SM) we present derivations and proofs of the results shown in the Letter, namely of the continuity equation for individual trajectories, the properties of initial-point currents and dual-reversed dynamics, results for the correlation functions and variance, as well as the bounds in the limit of small window size. Moreover, we comment on the one-dimensional case where second moments of empirical density and current are finite even for vanishing window size. In addition, we show details and supplementary figures related to results of large deviation theory. Finally, we list all information that is necessary to reproduce all simulations and analytical results shown in the figures presented the Letter.

CONTENTS

I. Continuity equation	1
II. Dual-reversal	2
A. Initial-point current from dual-reversal – direct proof	2
B. Proof of the dual-reversal symmetry	3
C. Equality of mean paths	6
D. Initial-point current from dual-reversal – proof via dual-reversal symmetry	6
III. Derivation of the mean, variance and correlation results	7
A. Mean of empirical density and current	7
B. Feynman-Kac approach	7
C. Dyson expansion	8
IV. Limit to delta function	10
A. Density variance	10
B. Correlation of current and density	12
C. Current variance	14
D. Limit to delta function in the one-dimensional case	15
V. Connection to Large Deviation Theory	16
VI. Numerical and analytical evaluation used for the figures	17
A. Analytical results for the two-dimensional Ornstein-Uhlenbeck process	18
B. Details and simulation parameters for figures in the Letter	19
References	19

I. CONTINUITY EQUATION

We define the empirical density and current coarse-grained over a window $U_{\mathbf{x}}(\mathbf{x}')$ as Eq. (1) in the Letter as

$$\begin{aligned}
 \overline{\rho^U}(\mathbf{x}, t) &\equiv \frac{1}{t} \int_0^t U_{\mathbf{x}}(\mathbf{x}_\tau) d\tau \\
 \overline{\mathbf{J}^U}(\mathbf{x}, t) &\equiv \frac{1}{t} \int_{\tau=0}^{\tau=t} U_{\mathbf{x}}(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau.
 \end{aligned} \tag{S1}$$

Consider a translation-invariant window function, $U_{\mathbf{x}}(\mathbf{x}') = U_0(\mathbf{x}' - \mathbf{x})$, e.g. a Gaussian or the indicator function of any norm ($U \propto \mathbf{1}_{\|\mathbf{x}-\mathbf{y}\| \leq h}$). Then $-\nabla_{\mathbf{x}} U_{\mathbf{x}}(\mathbf{x}') = \nabla_{\mathbf{x}'} U_{\mathbf{x}}(\mathbf{x}') \equiv \{\nabla U_{\mathbf{x}}\}(\mathbf{x}')$, yielding

$$\begin{aligned} -\nabla_{\mathbf{x}} \int_{\tau=0}^{\tau=t} U_{\mathbf{x}}(\mathbf{x}_{\tau}) \circ d\mathbf{x}_{\tau} &= \int_{\tau=0}^{\tau=t} \{\nabla U_{\mathbf{x}}\}(\mathbf{x}_{\tau}) \circ d\mathbf{x}_{\tau} \\ &= U_{\mathbf{x}}(\mathbf{x}_t) - U_{\mathbf{x}}(\mathbf{x}_0) \\ &= \partial_t \int_0^t U_{\mathbf{x}}(\mathbf{x}_{\tau}) d\tau, \end{aligned} \quad (\text{S2})$$

which proves the continuity equation, i.e. Eq. (3) in the Letter,

$$\partial_t \left[t \overline{\rho^U}(\mathbf{x}, t) \right] = -\nabla_{\mathbf{x}} \cdot t \overline{\mathbf{J}^U}(\mathbf{x}, t). \quad (\text{S3})$$

This is a continuity equation on the level of individual trajectories. For steady-state dynamics and normalized window functions $\int d^d z U_{\mathbf{x}}(\mathbf{z}) = 1$, taking the mean $\langle \cdot \rangle_s$ of this equation leads to a continuity equation for (coarse-grained) probability densities. Conversely, for non-normalized window functions $\int d^d z U_{\mathbf{x}}(\mathbf{z}) = \text{Volume}(U_{\mathbf{x}})$, the mean $\langle \cdot \rangle_s$ of Eq. (S3) can be interpreted as a continuity equation for probabilities.

Note that $\int_{\tau=0}^{\tau=t} [\{\nabla U_{\mathbf{x}}\}(\mathbf{x}_{\tau})] \circ d\mathbf{x}_{\tau} = U_{\mathbf{x}}(\mathbf{x}_t) - U_{\mathbf{x}}(\mathbf{x}_0)$ only holds for the Stratonovich integral and e.g. not for an Itô integral. Therefore, the continuity equation requires currents to be defined via Stratonovich integration, which is also required for the condition $\langle \overline{\mathbf{J}^U}(\mathbf{x}, t) \rangle_s = \mathbf{j}_s(\mathbf{x})$ (see Eqs. (S7) and (S41) and [1]) on the mean empirical current, and for consistency of time reversal [2].

II. DUAL-REVERSAL

A. Initial-point current from dual-reversal – direct proof

We consider the Itô Langevin equation

$$d\mathbf{x}_{\tau} = \mathbf{F}(\mathbf{x}_{\tau}) d\tau + \boldsymbol{\sigma} d\mathbf{W}_{\tau}, \quad (\text{S4})$$

which on the level of a transition probability density function corresponds to the Fokker-Planck equation

$$(\partial_t + \nabla_{\mathbf{x}} \cdot \hat{\mathbf{j}}_{\mathbf{x}}) \mathcal{G}(\mathbf{x}, t | \mathbf{y}) = 0, \quad (\text{S5})$$

with initial condition $\mathcal{G}(\mathbf{x}, 0 | \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ and current operator $\hat{\mathbf{j}}_{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - \mathbf{D} \nabla_{\mathbf{x}} = \mathbf{j}_s(\mathbf{x}) P_s(\mathbf{x})^{-1} - P_s(\mathbf{x}) \mathbf{D} \nabla_{\mathbf{x}} P_s(\mathbf{x})^{-1}$, with positive definite diffusion matrix $\mathbf{D} = \boldsymbol{\sigma} \boldsymbol{\sigma}^T / 2$. We assume that $\mathbf{F}(\mathbf{x})$ is sufficiently confining such that the system possesses a steady-state distribution $P_s(\mathbf{x}')$ and steady-state current $\mathbf{j}_s(\mathbf{x}')$.

The necessity of looking at dual-reversed dynamics emerges naturally when computing the initial-point current. The initial-point current in turn emerges when computing the statistics of current correlations, covariances and variances, since second moments depend on two-point functions, and for the current we need to compute the contributions from initial and final points of these two-point functions (e.g. compare Eq. (S60)).

We here give a direct proof that the initial-point current is obtained from the dual-reversed current operator $\hat{\mathbf{j}}_{\mathbf{x}}^{\dagger} = \hat{\mathbf{j}}_s(\mathbf{x}) - \hat{\mathbf{j}}_g(\mathbf{x})$ where $\hat{\mathbf{j}}_s(\mathbf{x}) = \mathbf{j}_s(\mathbf{x}) / P_s(\mathbf{x})$ and $\hat{\mathbf{j}}_g(\mathbf{x}) = -P_s(\mathbf{x}) \mathbf{D} \nabla_{\mathbf{x}} P_s(\mathbf{x})^{-1}$ are the parts of the Fokker-Planck current $\hat{\mathbf{j}}_{\mathbf{x}} = \hat{\mathbf{j}}_s(\mathbf{x}) + \hat{\mathbf{j}}_g(\mathbf{x})$. Later we will also show how this arises as a consequence of the dual-reversal symmetry.

We define the final-point current at \mathbf{y} as the mean displacement

$$\hat{\mathbf{j}}_{\mathbf{f}}(\mathbf{y}, t; \mathbf{x}, 0) dt \equiv \langle \delta(\mathbf{x} - \mathbf{x}_0) \delta(\mathbf{y} - \mathbf{x}_t) \circ d\mathbf{x}_t \rangle, \quad (\text{S6})$$

yielding

$$\begin{aligned} \hat{\mathbf{j}}_{\mathbf{f}}(\mathbf{y}, t; \mathbf{x}, 0) dt &\equiv \langle \delta(\mathbf{x} - \mathbf{x}_0) \delta(\mathbf{y} - \mathbf{x}_t) d\mathbf{x}_t \rangle + \frac{1}{2} \langle \delta(\mathbf{x} - \mathbf{x}_0) d[\delta(\mathbf{y} - \mathbf{x}_t)] d\mathbf{x}_t \rangle \\ &= [\mathbf{F}(\mathbf{y}) - \mathbf{D} \nabla_{\mathbf{y}}] P_{\mathbf{x}}(\mathbf{y}, t) dt, \end{aligned} \quad (\text{S7})$$

which agrees with $\hat{\mathbf{j}}_{\mathbf{y}} P_{\mathbf{x}}(\mathbf{y}, t) dt$ as required.

We similarly define the initial-point current at \mathbf{x} and calculate

$$\begin{aligned}\mathbf{j}_{\text{in}}(\mathbf{y}, t; \mathbf{x}, 0)dt &\equiv \langle \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{y} - \mathbf{x}_t) \circ d\mathbf{x}_0 \rangle \\ &= \langle \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{y} - \mathbf{x}_t)d\mathbf{x}_0 \rangle - \mathbf{D}\nabla_{\mathbf{x}}P_{\mathbf{x}}(\mathbf{y}, t)dt \\ &= \hat{\mathbf{j}}_{\mathbf{x}}P_{\mathbf{x}}(\mathbf{y}, t)dt + \langle \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{y} - \mathbf{x}_t)(d\mathbf{x}_0 - \langle d\mathbf{x}_0 \rangle) \rangle,\end{aligned}\quad (\text{S8})$$

We use $\boldsymbol{\varepsilon}$ for $d\mathbf{x}_0 = \mathbf{F}(\mathbf{x})dt + \sqrt{2D}d\mathbf{W}_0$ (for simplicity and without loss of generality we consider isotropic diffusion $\boldsymbol{\sigma} = \sqrt{2D}\mathbf{1}$; the proof for non-isotropic diffusion can be done similarly, or can be seen as a consequence of dual-reversal symmetry (see below)). Then we obtain, denoting a d -dimensional volume integration over $\boldsymbol{\varepsilon}$ by $\int d^d\boldsymbol{\varepsilon}$,

$$\langle \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{y} - \mathbf{x}_t)(d\mathbf{x}_0 - \langle d\mathbf{x}_0 \rangle) \rangle = P_{\mathbf{s}}(\mathbf{x}) \int d^d\boldsymbol{\varepsilon} \mathcal{G}(\mathbf{y}, t - dt|\mathbf{x} + \boldsymbol{\varepsilon}) \mathcal{G}(\mathbf{x} + \boldsymbol{\varepsilon}, dt|\mathbf{x}) [\boldsymbol{\varepsilon} - \mathbf{F}(\mathbf{x})dt], \quad (\text{S9})$$

where we expand

$$\begin{aligned}\mathcal{G}(\mathbf{y}, t - dt|\mathbf{x} + \boldsymbol{\varepsilon}) &\approx \mathcal{G}(\mathbf{y}, t - dt|\mathbf{x}) + \boldsymbol{\varepsilon} \cdot \nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t - dt|\mathbf{x}) \\ &\approx \mathcal{G}(\mathbf{y}, t|\mathbf{x}) - \partial_t\mathcal{G}(\mathbf{y}, t|\mathbf{x})dt + \boldsymbol{\varepsilon} \cdot \nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t|\mathbf{x}).\end{aligned}\quad (\text{S10})$$

The first two terms give zero since $\boldsymbol{\varepsilon} - \mathbf{F}(\mathbf{x})dt$ is symmetric around zero. The last term gives

$$\langle \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{y} - \mathbf{x}_t)(d\mathbf{x}_0 - \langle d\mathbf{x}_0 \rangle) \rangle = P_{\mathbf{s}}(\mathbf{x}) \int d^d\boldsymbol{\varepsilon} [\nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t|\mathbf{x}) \cdot \boldsymbol{\varepsilon}] \mathcal{G}(\mathbf{x} + \boldsymbol{\varepsilon}, dt|\mathbf{x}) [\boldsymbol{\varepsilon} - \mathbf{F}(\mathbf{x})dt]. \quad (\text{S11})$$

Since dt is small, we may use the short-time propagator [3]

$$\mathcal{G}(\mathbf{x} + \boldsymbol{\varepsilon}, dt|\mathbf{x}) \approx \mathcal{G}_{\text{short}}(\mathbf{x} + \boldsymbol{\varepsilon}, dt|\mathbf{x}) = (4\pi Ddt)^{-d/2} \exp\left[-\frac{[\boldsymbol{\varepsilon} - \mathbf{F}(\mathbf{x})dt]^2}{4Ddt}\right]. \quad (\text{S12})$$

Shifting $\boldsymbol{\varepsilon} - \mathbf{F}(\mathbf{x})dt = \mathbf{w}$ we get, using isotropy from the second to third line,

$$\begin{aligned}\langle \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{y} - \mathbf{x}_t)(d\mathbf{x}_0 - \langle d\mathbf{x}_0 \rangle) \rangle &= P_{\mathbf{s}}(\mathbf{x}) \int d^d\mathbf{w} \mathcal{G}_{\text{short}}(\mathbf{w}, dt|\mathbf{0}) [\nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t|\mathbf{x}) \cdot (\mathbf{w} + \mathbf{F}(\mathbf{x})dt)] \mathbf{w} \\ &= P_{\mathbf{s}}(\mathbf{x}) \int d^d\mathbf{w} \mathcal{G}_{\text{short}}(\mathbf{w}, dt|\mathbf{0}) [\nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t|\mathbf{x}) \cdot \mathbf{w}] \mathbf{w} \\ &= P_{\mathbf{s}}(\mathbf{x}) \nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t|\mathbf{x}) \int d^d\mathbf{w} \mathcal{G}_{\text{short}}(\mathbf{w}, dt|\mathbf{0}) w_1^2 \\ &= P_{\mathbf{s}}(\mathbf{x}) \nabla_{\mathbf{x}}\mathcal{G}(\mathbf{y}, t|\mathbf{x}) 2Ddt \\ &= -2\hat{\mathbf{j}}_g(\mathbf{x})P_{\mathbf{x}}(\mathbf{y}, t)dt.\end{aligned}\quad (\text{S13})$$

Overall we get that the initial point current follows from the dual-reversed current operator

$$\mathbf{j}_{\text{in}}(\mathbf{y}, t; \mathbf{x}, 0) = [\hat{\mathbf{j}}_{\mathbf{x}} - 2\hat{\mathbf{j}}_g(\mathbf{x})]P_{\mathbf{x}}(\mathbf{y}, t) \equiv \hat{\mathbf{j}}_{\mathbf{x}}^{\dagger}P_{\mathbf{x}}(\mathbf{y}, t). \quad (\text{S14})$$

This proves the second part of Eq. (9) in the Letter.

B. Proof of the dual-reversal symmetry

Let \mathbf{x}_t obey the Itô equation (S4). We recall the unique decomposition of the drift field $\mathbf{F}(\mathbf{x}) = \mathbf{D}\{\nabla_{\mathbf{x}} \ln P_{\mathbf{s}}(\mathbf{x})\} + P_{\mathbf{s}}^{-1}(\mathbf{x})\mathbf{j}_{\mathbf{s}}(\mathbf{x})$ [4, 5], where curly brackets $\{\cdot\}$ denote that derivatives act only within $\{\cdot\}$ but not further. The forward and backward Fokker-Planck equations for the Green's function read

$$[\partial_t + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) - \nabla_{\mathbf{x}} \cdot \mathbf{D}\nabla_{\mathbf{x}}]G(\mathbf{x}, t|\mathbf{x}') = 0, \quad [\partial_t - \mathbf{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} - \nabla_{\mathbf{x}} \cdot \mathbf{D}\nabla_{\mathbf{x}}]G(\mathbf{x}', t|\mathbf{x}) = 0, \quad (\text{S15})$$

with the initial condition $G(\mathbf{x}', 0|\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$ (for both) and appropriate boundary conditions respectively. Let the invariant density be $\lim_{t \rightarrow \infty} G(\mathbf{x}, t|\mathbf{x}') = P_{\mathbf{s}}(\mathbf{x})$.

According to [6] the *dual transition probability density* $G^{\dagger}(\mathbf{x}', t|\mathbf{x})$ is in general defined such that

$$G^{\dagger}(\mathbf{x}', t|\mathbf{x})P_{\mathbf{s}}(\mathbf{x}) = G(\mathbf{x}, t|\mathbf{x}')P_{\mathbf{s}}(\mathbf{x}') \quad (\text{S16})$$

holds. It does not need to relate to time-reversal in general. We will prove the following **dual-reversal symmetry**:

If $G(\mathbf{x}, t|\mathbf{x}')$ is the Green's function of the Itô diffusion \mathbf{x}_t defined in Eq. (S4) the dual symmetry in Eq. (S16) holds if and only if $G^\dagger(\mathbf{x}', t|\mathbf{x})$ is the Green's function of \mathbf{x}_t^\dagger – the time-reverse with simultaneous inversion of $\hat{\mathbf{j}}_s(\mathbf{x}) \rightarrow -\hat{\mathbf{j}}_s(\mathbf{x})$.

This symmetry can, as in Eq. (9) in the Letter, be expressed as

$$P_{\mathbf{x}}(\mathbf{y}, t) = P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t), \quad (\text{S17})$$

where $P_{\mathbf{x}}(\mathbf{y}, t) \equiv G(\mathbf{y}, t|\mathbf{x})P_s(\mathbf{x})$, and is thus a generalization of the detailed balance condition $P_{\mathbf{x}}(\mathbf{y}, t) = P_{\mathbf{y}}(\mathbf{x}, t)$, that holds for $\mathbf{j}_s = \mathbf{0}$.

We first prove the *necessary* (i.e. “only if”) part. Because we are considering a continuous Markov process the so-called Lindeberg condition is satisfied uniformly in $\mathbf{x}, \mathbf{x}', t, \delta t$ for any $\epsilon > 0$

$$\lim_{\delta t \rightarrow 0} \int_{|\mathbf{x} - \mathbf{x}'| > \epsilon} d^d x G(\mathbf{x}', t + \delta t|\mathbf{x}, t) = 0. \quad (\text{S18})$$

Moreover, since the process is assumed to be time-homogeneous $G(\mathbf{x}', t + \delta t|\mathbf{x}, t) = G(\mathbf{x}', \delta t|\mathbf{x}, 0) \equiv G(\mathbf{x}', \delta t|\mathbf{x})$.

The condition for Eq. (S16) to hold is a condition on $\mathbf{F}^\dagger(\mathbf{x})$ and (potentially) \mathbf{D}^\dagger . Thus, we begin by recalling the probabilistic definition of $\mathbf{F}(\mathbf{x})$ and \mathbf{D} (e.g. from the Champman-Kolmogorov equation). Namely, these are defined by the scaling limits

$$\lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{x} - \mathbf{x}'| < \epsilon} d^d x (\mathbf{x}'_i - \mathbf{x}_i) G(\mathbf{x}', t + \delta t|\mathbf{x}, t) \equiv \mathbf{F}_i(\mathbf{x}), \quad (\text{S19})$$

for each component $\mathbf{F}_i(\mathbf{x})$, and

$$\lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{x} - \mathbf{x}'| < \epsilon} d^d x (\mathbf{x}'_i - \mathbf{x}_i)(\mathbf{x}'_j - \mathbf{x}_j) G(\mathbf{x}', t + \delta t|\mathbf{x}, t) \equiv 2\mathbf{D}_{ij}, \quad (\text{S20})$$

for the diffusion matrix, where we have tacitly used that we consider additive noise diffusion only (generalizations to multiplicative noise are straightforward but are not required here). Higher-order coefficients vanish. To see why we consider the third-order quantity [7]

$$\lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{x} - \mathbf{x}'| < \epsilon} d^d x (\mathbf{x}'_i - \mathbf{x}_i)(\mathbf{x}'_j - \mathbf{x}_j)(\mathbf{x}'_k - \mathbf{x}_k) G(\mathbf{x}', t + \delta t|\mathbf{x}, t) \equiv C_{ijk}(\mathbf{x}) + \mathcal{O}(\epsilon). \quad (\text{S21})$$

It follows from the definition in Eq. (S21) that $C_{ijk}(\mathbf{x})$ is symmetric in i, j, k . Let us introduce $C(\boldsymbol{\delta}, \mathbf{x}, t) \equiv \sum_{i,j,k} C_{ijk}(\mathbf{x}) \delta_i \delta_j \delta_k$ for some $|\boldsymbol{\delta}| > 0$, such that in turn $C_{ijk}(\mathbf{x}) = \partial_{\delta_i}^3 \partial_{\delta_j} \partial_{\delta_k} C(\boldsymbol{\delta}, \mathbf{x}, t)/3!$. Then we have using Eq. (S21)

$$\begin{aligned} |C(\boldsymbol{\delta}, \mathbf{x}, t)| &= \lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \left| \int_{|\mathbf{x} - \mathbf{x}'| < \epsilon} \boldsymbol{\delta} \cdot (\mathbf{x}' - \mathbf{x}) [\boldsymbol{\delta} \cdot (\mathbf{x}' - \mathbf{x})]^2 G(\mathbf{x}', t + \delta t|\mathbf{x}, t) d^d x \right| + \mathcal{O}(\epsilon) \\ &\leq \lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{x} - \mathbf{x}'| < \epsilon} |\boldsymbol{\delta} \cdot (\mathbf{x}' - \mathbf{x})| [\boldsymbol{\delta} \cdot (\mathbf{x}' - \mathbf{x})]^2 G(\mathbf{x}', t + \delta t|\mathbf{x}, t) d^d x + \mathcal{O}(\epsilon) \\ &\leq \epsilon |\boldsymbol{\delta}| \lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{x} - \mathbf{x}'| < \epsilon} [\boldsymbol{\delta} \cdot (\mathbf{x}' - \mathbf{x})]^2 G(\mathbf{x}', t + \delta t|\mathbf{x}, t) d^d x + \mathcal{O}(\epsilon) \\ &= \epsilon |\boldsymbol{\delta}| \left[\sum_{ij} 2\mathbf{D}_{ij} \delta_i \delta_j + \mathcal{O}(\epsilon) \right] + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon), \end{aligned} \quad (\text{S22})$$

which by Eq. (S21) implies $C_{ijk}(\mathbf{x}) = 0$. One can analogously show that all higher-order coefficients vanish as well.

Let us now set $\mathbf{x}' = \mathbf{x} + \mathbf{h}$. If Eq. (S16) is to hold then also

$$\lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{h}| < \epsilon} d^d h \mathbf{h}_i G^\dagger(\mathbf{x} + \mathbf{h}, t|\mathbf{x}) P_s(\mathbf{x}) = \lim_{\substack{\delta t \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{h}| < \epsilon} d^d h \mathbf{h}_i G(\mathbf{x}, t|\mathbf{x} + \mathbf{h}) P_s(\mathbf{x} + \mathbf{h}) \quad (\text{S23})$$

must hold. The left hand side of Eq. (S23) yields simply (see Eq. (S19)) $\mathbf{F}_i^\dagger(\mathbf{x})$. The right hand side is somewhat trickier, and we proceed as follows. We write

$$\begin{aligned} G(\mathbf{x}, t|\mathbf{x} + \mathbf{h}) P_s(\mathbf{x} + \mathbf{h}) &= G(\mathbf{z} - \mathbf{h} + \mathbf{h}, t|\mathbf{x} + \mathbf{h}) P_s(\mathbf{x} + \mathbf{h}) \quad (\equiv f(\mathbf{x} + \mathbf{h})) \\ &= G(\mathbf{x} - \mathbf{h}, t|\mathbf{x}) P_s(\mathbf{x}) + \mathbf{h} \cdot \nabla [G(\mathbf{x} - \mathbf{h}, t|\mathbf{x}) P_s(\mathbf{x})] + \mathcal{O}(\mathbf{h}^2), \end{aligned} \quad (\text{S24})$$

and therefore

$$\begin{aligned} \lim_{\substack{\delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{h}| < \varepsilon} d^d h \mathbf{h}_i G(\mathbf{x}, t | \mathbf{x} + \mathbf{h}) P_s(\mathbf{x} + \mathbf{h}) \\ = -\mathbf{F}_i(\mathbf{x}) P_s(\mathbf{x}) + 2 \sum_j \mathbf{D}_{ij} \partial_{x_j} P_s(\mathbf{x}), \end{aligned} \quad (\text{S25})$$

where we have used Eqs. (S19-S20) and the minus sign is a result of a change of integration variable. Eq. (S25) is obviously the “backward drift”. Therefore, a necessary condition for Eq. (S16) to hold is (written suggestively)

$$P_s(\mathbf{x}) \mathbf{F}_i^\dagger(\mathbf{x}) = -P_s(\mathbf{x}) \mathbf{F}_i(\mathbf{x}) + 2 \sum_j \mathbf{D}_{ij} \partial_{x_j} P_s(\mathbf{x}). \quad (\text{S26})$$

Since the dual dynamics by construction must have the same invariant density and because the decomposition of any $\mathbf{F}(\mathbf{x})$ into mutually orthogonal conservative and incompressible components is general for any time-homogeneous ergodic Itô diffusion we must have (note that \mathbf{F} and \mathbf{F}^\dagger are vectors not operators and thus commute with $P_s(\mathbf{x})$), $\mathbf{F}^\dagger(\mathbf{x}) = \mathbf{D} \nabla_{\mathbf{x}} \ln P_s(\mathbf{x}) + \mathbf{Z}^\dagger(\mathbf{x})$ (where $\mathbf{Z}^\dagger(\mathbf{x}) \cdot \mathbf{D} \nabla_{\mathbf{x}} \ln P_s(\mathbf{x}) = 0$) and therefore

$$\sum_j \mathbf{D}_{ij} \partial_{x_j} P_s(\mathbf{x}) + \mathbf{Z}_i^\dagger(\mathbf{x}) P_s(\mathbf{x}) = - \sum_j \mathbf{D}_{ij} \partial_{x_j} P_s(\mathbf{x}) - (\hat{\mathbf{j}}_s)_i(\mathbf{x}) + 2 \sum_j \mathbf{D}_{ij} \partial_{x_j} P_s(\mathbf{x}), \quad (\text{S27})$$

and it is now not difficult to see that $G^\dagger(\mathbf{x}, t | \mathbf{x}')$ – the Green’s function of some forward Fokker-Planck equation (to be determined below) – has invariant density $P_s(\mathbf{x})$ if we have $\mathbf{Z}(\mathbf{x})^\dagger P_s(\mathbf{x}) = -\hat{\mathbf{j}}_s(\mathbf{x})$ which solves Eq. (S27).

A straightforward calculation along the same lines shows that

$$\lim_{\substack{\delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{h}| < \varepsilon} d^d h \mathbf{h}_i \mathbf{h}_j G^\dagger(\mathbf{x} + \mathbf{h}, t | \mathbf{x}) P_s(\mathbf{x}) = \lim_{\substack{\delta t \rightarrow 0 \\ \varepsilon \rightarrow 0}} \frac{1}{\delta t} \int_{|\mathbf{h}| < \varepsilon} d^d h \mathbf{h}_i \mathbf{h}_j G(\mathbf{x}, t | \mathbf{x} + \mathbf{h}) P_s(\mathbf{x} + \mathbf{h}) \quad (\text{S28})$$

leads to the trivial condition $\mathbf{D}^\dagger = \mathbf{D}$. Moreover, higher orders in \mathbf{h} vanish as in Eq. (S22). Therefore, the necessary condition for the dual $G^\dagger(\mathbf{x}, t | \mathbf{x}')$ to (i) have the same invariant density $P_s(\mathbf{x})$ and (ii) for Eq. (S16) to hold is that

$$\mathbf{F}^\dagger(\mathbf{x}) = \{\mathbf{D} \nabla_{\mathbf{x}} \ln P_s(\mathbf{x})\} - P_s^{-1}(\mathbf{x}) \hat{\mathbf{j}}_s(\mathbf{x}). \quad (\text{S29})$$

Furthermore, if $\hat{L}_{\mathbf{x}}$ is the corresponding forward Fokker-Planck operator with drift \mathbf{F}^\dagger for $G^\dagger(\mathbf{x}, t | \mathbf{x}')$ then $\hat{L}_{\mathbf{x}}^\dagger$ is the corresponding backward Fokker-Planck operator for $G^\dagger(\mathbf{x}', t | \mathbf{x})$. We now show that the necessary condition Eq. (S29) is also sufficient.

To do so we write

$$G^\dagger(\mathbf{x}, t | \mathbf{x}') = G(\mathbf{x}', t | \mathbf{x}) \frac{P_s(\mathbf{x})}{P_s(\mathbf{x}')}, \quad (\text{S30})$$

and assume that condition Eq. (S29) holds. The left and right hand side of Eq. (S30) have the same initial condition, $G^\dagger(\mathbf{x}, 0 | \mathbf{x}') = G(\mathbf{x}', 0 | \mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$. Plugging $G^\dagger(\mathbf{x}, t | \mathbf{x}')$ in Eq. (S30) into the forward Fokker-Planck equation $\partial_t G^\dagger(\mathbf{x}, t | \mathbf{x}') = (-\nabla_{\mathbf{x}} \cdot \mathbf{F}^\dagger(\mathbf{x}) + \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}}) G^\dagger(\mathbf{x}, t | \mathbf{x}')$ we obtain

$$-\nabla_{\mathbf{x}} \cdot \mathbf{F}^\dagger(\mathbf{x}) G^\dagger(\mathbf{x}', t | \mathbf{x}) = -[G(\mathbf{x}', t | \mathbf{x}) \nabla_{\mathbf{x}} \cdot \mathbf{F}^\dagger(\mathbf{x}) P_s(\mathbf{x}) + \mathbf{F}^\dagger(\mathbf{x}) P_s(\mathbf{x}) \cdot \nabla_{\mathbf{x}} G(\mathbf{x}', t | \mathbf{x})] P_s^{-1}(\mathbf{x}') \quad (\text{S31})$$

for the drift term, and

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} G^\dagger(\mathbf{x}', t | \mathbf{x}) &= \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} G(\mathbf{x}', t | \mathbf{x}) \frac{P_s(\mathbf{x})}{P_s(\mathbf{x}')} \\ &= \left[P_s(\mathbf{x}) \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} G(\mathbf{x}', t | \mathbf{x}) + 2 \sum_{ij} D_{ij} \partial_{x_i} P_s(\mathbf{x}) \partial_{x_j} G(\mathbf{x}', t | \mathbf{x}) + G(\mathbf{x}', t | \mathbf{x}) \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} P_s(\mathbf{x}) \right] P_s^{-1}(\mathbf{x}'). \end{aligned} \quad (\text{S32})$$

Note that the first term in Eq. (S31) and the last term in Eq. (S31) sum to zero since $P_s(\mathbf{x})$ is the stationary solution of the forward Fokker-Planck equation, i.e.

$$G(\mathbf{x}', t | \mathbf{x}) [-\nabla_{\mathbf{x}} \cdot \mathbf{F}^\dagger(\mathbf{x}) P_s(\mathbf{x}) + \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} P_s(\mathbf{x})] P_s^{-1}(\mathbf{x}') = 0, \quad (\text{S33})$$

and the remaining terms yield

$$\frac{P_s(\mathbf{x})}{P_s(\mathbf{x}')} \left[-\mathbf{F}^\dagger(\mathbf{x}) \cdot \nabla_{\mathbf{x}} G(\mathbf{x}', t|\mathbf{x}) + 2 \sum_{ij} D_{ij} \partial_{x_i} \ln P_s(\mathbf{x}) \partial_{x_j} G(\mathbf{x}', t|\mathbf{x}) + \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} G(\mathbf{x}', t|\mathbf{x}) \right]. \quad (\text{S34})$$

We now substitute Eq. (S29) and rearrange terms to obtain

$$\frac{P_s(\mathbf{x})}{P_s(\mathbf{x}')} [\mathbf{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} G(\mathbf{x}', t|\mathbf{x}) + \nabla_{\mathbf{x}} \cdot \mathbf{D} \nabla_{\mathbf{x}} G(\mathbf{x}', t|\mathbf{x})] = \partial_t G(\mathbf{x}', t|\mathbf{x}) \frac{P_s(\mathbf{x})}{P_s(\mathbf{x}')}, \quad (\text{S35})$$

where in the last step we identified the backward Fokker-Planck equation in Eq. (S15). Thus

$$\partial_t G^\dagger(\mathbf{x}, t|\mathbf{x}') = \partial_t G(\mathbf{x}', t|\mathbf{x}) \frac{P_s(\mathbf{x})}{P_s(\mathbf{x}')}, \quad (\text{S36})$$

and since $G^\dagger(\mathbf{x}, t|\mathbf{x}')$ and $G(\mathbf{x}', t|\mathbf{x})$ have the same initial condition, and since the solutions of the respective Fokker-Planck equations are unique this shows that the duality in Eq. (S30) holds at all times and we have thus proven that the condition Eq. (S29) is sufficient. This completes the proof of necessary and sufficient conditions for the dual-reverse of a time-homogeneous ergodic (additive noise) Itô diffusion. Thus, we have proven the dual reversal-symmetry in the first part of Eq. (9) in the Letter.

C. Equality of mean paths

We see in the Letter in Fig. 2a that the mean paths of forward- and dual-reversed dynamics between fixed points overlap. This can be seen as a consequence of the dual-reversal symmetry $P_{\mathbf{y} \rightarrow \mathbf{j}_s}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) = P_{\mathbf{x}}(\mathbf{y}, t)$ or equivalently $\mathcal{G}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t|\mathbf{y}) P_s(\mathbf{y}) = \mathcal{G}(\mathbf{y}, t|\mathbf{x}) P_s(\mathbf{x})$ as we show here. Consider $0 < \tau < t$, then the point \mathbf{z}_τ on the mean path $\mathbf{x} \rightarrow \mathbf{y}$ is given by an integral over all possible intermediate points $\mathbf{z}_\tau = \mathbf{z}'$ weighted by $\mathcal{G}(\mathbf{y}, t-\tau|\mathbf{z}') \mathcal{G}(\mathbf{z}', \tau|\mathbf{x}) / \mathcal{G}(\mathbf{y}, t|\mathbf{x})$ which gives the Chapman-Kolmogorov equation-like expression

$$\mathcal{G}(\mathbf{y}, t|\mathbf{x}) \mathbf{z}_\tau = \int d^d z' \mathcal{G}(\mathbf{y}, t-\tau|\mathbf{z}') \mathcal{G}(\mathbf{z}', \tau|\mathbf{x}) \mathbf{z}'. \quad (\text{S37})$$

The corresponding point on the mean dual-reversed path is given by

$$\begin{aligned} \mathcal{G}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t|\mathbf{y}) \mathbf{z}_\tau^\dagger &= \int d^d z' \mathcal{G}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, \tau|\mathbf{z}') \mathcal{G}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{z}', t-\tau|\mathbf{y}) \mathbf{z}' \\ &= \int d^d z' \mathcal{G}(\mathbf{z}', \tau|\mathbf{x}) \frac{P_s(\mathbf{x})}{P_s(\mathbf{z}')} \mathcal{G}(\mathbf{y}, t-\tau|\mathbf{z}') \frac{P_s(\mathbf{z}')}{P_s(\mathbf{y})} \mathbf{z}' \\ &= \frac{P_s(\mathbf{x})}{P_s(\mathbf{y})} \mathcal{G}(\mathbf{y}, t|\mathbf{x}) \mathbf{z}_\tau \\ &= \mathcal{G}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t|\mathbf{y}) \mathbf{z}_\tau, \end{aligned} \quad (\text{S38})$$

which implies $\mathbf{z}_\tau = \mathbf{z}_\tau^\dagger$ for all $0 < \tau < t$, so the mean paths indeed agree.

D. Initial-point current from dual-reversal – proof via dual-reversal symmetry

From the tangents to the mean paths (see also arrows in Fig. 2a-d in the Letter) we see that the initial-point current is the reflected final-point current of the dual dynamics,

$$\begin{aligned} \mathbf{j}_{\text{in}}(\mathbf{y}, t; \mathbf{x}, 0) &= -\hat{\mathbf{j}}_{\text{fi}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t; \mathbf{y}, 0) \\ &= -\hat{\mathbf{j}}_{\mathbf{x}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s} P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) \\ &= -[\hat{\mathbf{j}}_g(\mathbf{x}) - \hat{\mathbf{j}}_s(\mathbf{x})] P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) \\ &= \hat{\mathbf{j}}_{\mathbf{x}}^\dagger P_{\mathbf{x}}(\mathbf{y}, t), \end{aligned} \quad (\text{S39})$$

where we used the dual identity $P_{\mathbf{y}}^{\mathbf{j}_s \rightarrow -\mathbf{j}_s}(\mathbf{x}, t) = P_{\mathbf{x}}(\mathbf{y}, t)$ and the definition $\hat{\mathbf{j}}_{\mathbf{x}}^\dagger = \hat{\mathbf{j}}_s(\mathbf{x}) - \hat{\mathbf{j}}_g(\mathbf{x})$. This gives an alternative to the above proof for the second part of Eq. (9) in the main text

III. DERIVATION OF THE MEAN, VARIANCE AND CORRELATION RESULTS

In this section we derive Eqs. (7), (10) and (11) in the Letter.

A. Mean of empirical density and current

Expressions for the mean of the empirical density and current are mentioned between Eqs. (4) and (5) in the Letter. They can either be obtained by the Feynman-Kac approach (see below) or directly from

$$\begin{aligned}\left\langle \overline{\rho^U}(\mathbf{x}, t) \right\rangle_s &= \left\langle \frac{1}{t} \int_0^t U_{\mathbf{x}}(\mathbf{x}_\tau) d\tau \right\rangle_s \\ &= \frac{1}{t} \int_0^t d\tau \int d^d z U_{\mathbf{x}}(\mathbf{z}) P_s(\mathbf{z}) \\ &= \int d^d z U_{\mathbf{x}}(\mathbf{z}) P_s(\mathbf{z}),\end{aligned}\tag{S40}$$

and using the Itô-Stratonovich correction term and $d\mathbf{x}_\tau d\mathbf{x}_\tau^T = 2\mathbf{D}d\tau$ (where $\mathbf{D} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T/2$)

$$\begin{aligned}\left\langle \overline{\mathbf{J}^U}(\mathbf{x}, t) \right\rangle_s &= \left\langle \frac{1}{t} \int_0^t U_{\mathbf{x}}(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau \right\rangle_s \\ &= \frac{1}{t} \int_{\tau=0}^{\tau=t} \langle U_{\mathbf{x}}(\mathbf{x}_\tau) d\mathbf{x}_\tau \rangle_s + \frac{1}{t} \int_{\tau=0}^{\tau=t} \frac{1}{2} \langle dU_{\mathbf{x}}(\mathbf{x}_\tau) d\mathbf{x}_\tau \rangle_s \\ &= \int d^d z P_s(\mathbf{z}) [U_{\mathbf{x}}(\mathbf{z}) \mathbf{F}(\mathbf{z}) + \mathbf{D} \{ \nabla_{\mathbf{z}} U_{\mathbf{x}}(\mathbf{z}) \}] d\tau \\ &= \int d^d z U_{\mathbf{x}}(\mathbf{z}) [\mathbf{F}(\mathbf{z}) - \mathbf{D} \nabla_{\mathbf{z}}] P_s(\mathbf{z}) \\ &= \int d^d z U_{\mathbf{x}}(\mathbf{z}) \mathbf{j}_s(\mathbf{z}).\end{aligned}\tag{S41}$$

Note that if we had assumed an Itô integral instead, we would miss the ∇ -term.

B. Feynman-Kac approach

Let us define time-integrated current and density as

$$\begin{aligned}\mathbf{I}_t &= \int_{\tau=0}^{\tau=t} U(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau \\ R_t &= \int_0^t V(\mathbf{x}_\tau) d\tau,\end{aligned}\tag{S42}$$

then,

$$\begin{aligned}d\mathbf{I}_\tau &= U(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau = U(\mathbf{x}_\tau) d\mathbf{x}_\tau + \mathbf{D} \{ \nabla U \}(\mathbf{x}_\tau) d\tau \\ dR_\tau &= V(\mathbf{x}_\tau) d\tau.\end{aligned}\tag{S43}$$

We will derive an equation for the characteristic function of the joint distribution of $\mathbf{x}_t, R_t, \mathbf{I}_t$ via a Feynman-Kac approach which will then yield the moments (including variances and correlations) via a Dyson series. This formalism was already applied to the empirical density (under the term of local/occupation time (fraction)) [8–10].

We use Itô's Lemma in d dimensions for a test function $f = f(\mathbf{x}_t, R_t, \mathbf{I}_t)$ and Eqs. (S4) and (??),

$$\begin{aligned}df &= \sum_{i=1}^d \frac{\partial f}{\partial x_i} dx_t^i + \frac{\partial f}{\partial R} dR_t + \sum_{i=1}^d \frac{\partial f}{\partial I_i} dI_t^i + \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2 f}{\partial x_i \partial x_j} dx_t^i dx_t^j + \frac{\partial^2 f}{\partial I_i \partial I_j} dI_t^i dI_t^j + 2 \frac{\partial^2 f}{\partial x_i \partial I_j} dx_t^i dI_t^j \right) \\ &= [(\nabla_{\mathbf{x}} f) + (\nabla_I f) U(\mathbf{x}_t)] [\mathbf{F}(\mathbf{x}_t) dt + \boldsymbol{\sigma} d\mathbf{W}_t] + (\nabla_I f) \mathbf{D} \{ \nabla_{\mathbf{x}} U(\mathbf{x}_t) \} dt + V(\mathbf{x}_t) \partial_R f dt \\ &\quad + (\nabla_{\mathbf{x}}^T \mathbf{D} \nabla_{\mathbf{x}} + U(\mathbf{x}_t)^2 \nabla_I^T \mathbf{D} \nabla_I + 2U(\mathbf{x}_t) \nabla_{\mathbf{x}}^T \mathbf{D} \nabla_I) f dt.\end{aligned}\tag{S44}$$

Dividing by dt gives

$$\frac{d}{dt}f(\mathbf{x}_t, R_t, \mathbf{I}_t) = \left[\left(\mathbf{F} + \boldsymbol{\sigma} \frac{d\mathbf{W}_t}{dt} \right) (\nabla_{\mathbf{x}} + U\nabla_I) + \{\nabla_{\mathbf{x}}U\} \mathbf{D}\nabla_I + V\partial_R + \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_{\mathbf{x}} + U^2 \nabla_I^T \mathbf{D}\nabla_I + 2U \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_I \right] f. \quad (\text{S45})$$

Following this formalism, we move towards a Fokker-Planck equation. Employing the conditional probability density $Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0)$ we write

$$\begin{aligned} \frac{d}{dt}\langle f(\mathbf{x}_t, R_t, \mathbf{I}_t) \rangle &= \int d^d x \int_0^\infty dR \int d^d I f(\mathbf{x}, R, \mathbf{I}) \partial_t Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0) \\ &\stackrel{\text{It}\hat{0}}{=} \int d^d x \int_0^\infty dR \int d^d I Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0) \\ &\quad \left[\mathbf{F}(\nabla_{\mathbf{x}} + U\nabla_I) + \{\nabla_{\mathbf{x}}U\} \mathbf{D}\nabla_I + V\partial_R + \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_{\mathbf{x}} + U^2 \nabla_I^T \mathbf{D}\nabla_I + 2U \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_I \right] f(\mathbf{x}, R, \mathbf{I}) \\ &\stackrel{\text{IBP}}{=} \int d^d x \int_0^\infty dR \int d^d I f(\mathbf{x}, R, \mathbf{I}) \left[-\nabla_{\mathbf{x}} \mathbf{F} - U \mathbf{F} \nabla_I - \{\nabla_{\mathbf{x}}U\} \mathbf{D}\nabla_I - V\partial_R + \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_{\mathbf{x}} \right. \\ &\quad \left. + U^2 \nabla_I^T \mathbf{D}\nabla_I + 2U \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_I \right] Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0). \end{aligned} \quad (\text{S46})$$

Thus the Fokker-Planck equation reads

$$\partial_t Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0) = \hat{\mathcal{L}}_{x,R,I} Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0) \quad (\text{S47})$$

with the “tilted” Fokker-Planck operator (for discussion of the term $V\delta(R)$ entering at $R=0$ see [8])

$$\begin{aligned} \hat{\mathcal{L}}_{x,R,I} &= \hat{L} - U \mathbf{F} \cdot \nabla_I - \{\nabla_{\mathbf{x}}U\}^T \mathbf{D}\nabla_I - V\partial_R - V\delta(R) + U^2 \nabla_I^T \mathbf{D}\nabla_I + 2 \nabla_I^T \mathbf{D}\nabla_{\mathbf{x}} U \\ \hat{L} &= -\nabla_{\mathbf{x}} \cdot \mathbf{F} + \nabla_{\mathbf{x}}^T \mathbf{D}\nabla_{\mathbf{x}}. \end{aligned} \quad (\text{S48})$$

Now take a one dimensional Laplace variable v and a d -dimensional Fourier variable $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$ and Laplace and Fourier transform $Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0)$

$$\tilde{Q}_t(\mathbf{x}, v, \boldsymbol{\omega}|\mathbf{x}_0) = \int_0^\infty dR \int d^d I Q_t(\mathbf{x}, R, \mathbf{I}|\mathbf{x}_0) \exp(-vR - i\boldsymbol{\omega} \cdot \mathbf{I}). \quad (\text{S49})$$

C. Dyson expansion

Let the Fokker-Planck operator $\hat{L}(\mathbf{x})$ and the current operator $\hat{\mathbf{j}}_{\mathbf{x}}$ be defined via

$$\dot{\mathcal{G}}(\mathbf{x}, t|\mathbf{x}_0) = \hat{L}(\mathbf{x}) \mathcal{G}(\mathbf{x}, t|\mathbf{x}_0) = -\nabla_{\mathbf{x}}(\mathbf{F}(\mathbf{x}) - \mathbf{D}\nabla_{\mathbf{x}}) \mathcal{G}(\mathbf{x}, t|\mathbf{x}_0) = -\nabla_{\mathbf{x}} \cdot \hat{\mathbf{j}}_{\mathbf{x}} \mathcal{G}(\mathbf{x}, t|\mathbf{x}_0). \quad (\text{S50})$$

The Fourier-Laplace transformed tilted Fokker-Planck operator reads

$$\begin{aligned} \hat{\mathcal{L}}(\mathbf{x}, v, \boldsymbol{\omega}) &= \hat{L}(\mathbf{x}) - vV(\mathbf{x}) - i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}) - U(\mathbf{x})^2 \boldsymbol{\omega}^T \mathbf{D}\boldsymbol{\omega} \\ \hat{\mathbf{L}}_U(\mathbf{x}) &\stackrel{\text{def}}{=} U(\mathbf{x}) \hat{\mathbf{j}}_{\mathbf{x}} - \mathbf{D}\nabla_{\mathbf{x}} U(\mathbf{x}). \end{aligned} \quad (\text{S51})$$

We now restrict our attention to dynamics starting in the steady state P_s . Extensions of the formalism to any initial distribution are straightforward and introduce additional transient terms.

The moment generating function/characteristic function reads

$$\tilde{\mathcal{P}}_t^{\text{RI}}(v, \boldsymbol{\omega}|P_s) \equiv \langle e^{-vR_t - i\boldsymbol{\omega} \cdot \mathbf{I}_t} \rangle_s = 1 - v \langle R_t \rangle_s - i\boldsymbol{\omega} \cdot \langle \mathbf{I}_t \rangle_s + iv\boldsymbol{\omega} \langle R_t \mathbf{I}_t \rangle_s + O(\omega^2, v^2). \quad (\text{S52})$$

A Dyson expansion gives

$$\begin{aligned} e^{\hat{\mathcal{L}}(\mathbf{x}_1, v, \boldsymbol{\omega})t} &= 1 + \int_0^t dt_1 e^{\hat{L}(\mathbf{x}_1)(t-t_1)} \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] e^{\hat{L}(\mathbf{x}_1)t_1} \\ &+ \int_0^t dt_1 \int_0^{t_1} dt_2 e^{\hat{L}(\mathbf{x}_1)(t-t_1)} \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] e^{\hat{L}(\mathbf{x}_1)(t_1-t_2)} \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] e^{\hat{L}(\mathbf{x}_1)t_2} + O(\omega^2, v^2). \end{aligned} \quad (\text{S53})$$

The Dyson expansion allows us to obtain an expansion of the characteristic function (also see [8]). Using that the first propagation only differs to $\mathbf{1}$ by total derivatives, and using for the last propagation term $e^{\hat{L}(\mathbf{x}_1)t_2}P_s(\mathbf{x}_1) = P_s(\mathbf{x}_1)$, we obtain

$$\begin{aligned}\tilde{P}_t^{RI}(v, \boldsymbol{\omega}|P_s) &= \int d^d x_1 e^{\hat{L}(\mathbf{x}_1, v, \boldsymbol{\omega})t} P_s(\mathbf{x}_1) \\ &= 1 + \int d^d x_1 \int_0^t dt_1 \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] P_s(\mathbf{x}_1) \\ &+ \sum_{l,m=1}^d \int d^d x_1 \int_0^t dt_1 \int_0^{t_1} dt_2 \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] e^{\hat{L}(\mathbf{x}_1)(t_1-t_2)} \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] P_s(\mathbf{x}_1) + O(\omega^2, v^2).\end{aligned}\tag{S54}$$

Since $\mathcal{G}(\mathbf{x}_2, t|\mathbf{x}_1) = e^{\hat{L}(\mathbf{x}_1)t}\delta(\mathbf{x}_1 - \mathbf{x}_2)$ we can replace the propagation

$$\int d^d x_1 f(\mathbf{x}_1) e^{\hat{L}(\mathbf{x}_1)(t_1-t_2)} g(\mathbf{x}_1) = \int d^d x_1 \int d^d x_2 f(\mathbf{x}_2) \mathcal{G}(\mathbf{x}_2, t_1 - t_2|\mathbf{x}_1) g(\mathbf{x}_1),\tag{S55}$$

which gives

$$\begin{aligned}\tilde{P}_t^{RI}(v, \boldsymbol{\omega}|P_s) &= 1 + \int d^d x_1 \int_0^t dt_1 \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] P_s(\mathbf{x}_1) + \sum_{l,m=1}^d \int d^d x_1 \int d^d x_2 \int_0^t dt_1 \int_0^{t_1} dt_2 \\ &\left[vV(\mathbf{x}_2) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_2) \right] \mathcal{G}(\mathbf{x}_2, t_1 - t_2|\mathbf{x}_1) \left[vV(\mathbf{x}_1) + i\boldsymbol{\omega}^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}_1) \right] P_s(\mathbf{x}_1) + O(\omega^2, v^2).\end{aligned}\tag{S56}$$

Comparing the definition of the characteristic function Eq. (S52) with the result Eq. (S56) from the Dyson expansion, we obtain the moments and correlations of the empirical density $\bar{\rho}^U(\mathbf{y}, t)$ and current $\bar{\mathbf{J}}^U(\mathbf{x}, t)$.

The first moments give the same expressions as obtained in Eqs. (S40) and (S41). For second moments we use $U = U_{\mathbf{x}}$, $V = U_{\mathbf{y}}$ and

$$\frac{1}{t^2} \int_0^t dt_1 \int_0^{t_1} dt_2 f(t_1 - t_2) = \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) f(t'),\tag{S57}$$

and the notation (for convenience we change $\mathbf{x}_1, \mathbf{x}_2 \rightarrow \mathbf{z}, \mathbf{z}'$)

$$\hat{\mathcal{I}}_{\mathbf{xy}}^t[\cdot] = \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{y}}(\mathbf{z}')[\cdot],\tag{S58}$$

to obtain the known result [8, 10],

$$\left\langle \bar{\rho}^U(\mathbf{x}, t) \bar{\rho}^U(\mathbf{y}, t) \right\rangle_s = \hat{\mathcal{I}}_{\mathbf{xy}}^t [P_{\mathbf{z}'}(\mathbf{z}, t') + P_{\mathbf{z}}(\mathbf{z}', t')],\tag{S59}$$

which is shown in Eq. (7) in the Letter.

Integrating by parts and using that all quantities vanish at infinity, the correlation of R_t and \mathbf{I}_t can be written as

$$\begin{aligned}\left\langle \bar{\mathbf{J}}^U(\mathbf{x}, t) \bar{\rho}^U(\mathbf{y}, t) \right\rangle_s &= \frac{1}{t^2} \langle \mathbf{I}_t R_t \rangle_s \\ &= \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) \int d^d z \int d^d z' \left[\hat{\mathbf{L}}_{U_{\mathbf{x}}}(\mathbf{z}) \mathcal{G}(\mathbf{z}, t'|\mathbf{z}') U_{\mathbf{y}}(\mathbf{z}') + U_{\mathbf{y}}(\mathbf{z}) \mathcal{G}(\mathbf{z}, t'|\mathbf{z}') \hat{\mathbf{L}}_{U_{\mathbf{x}}}(\mathbf{z}') \right] P_s(\mathbf{z}') \\ &= \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) \int d^d z \int d^d z' \left[U_{\mathbf{x}}(\mathbf{z}) \hat{\mathbf{j}}_{\mathbf{z}} \mathcal{G}(\mathbf{z}, t'|\mathbf{z}') U_{\mathbf{y}}(\mathbf{z}') + U_{\mathbf{y}}(\mathbf{z}) \mathcal{G}(\mathbf{z}, t'|\mathbf{z}') [U_{\mathbf{x}}(\mathbf{z}') \hat{\mathbf{j}}_{\mathbf{z}'} - \mathbf{D} \nabla_{\mathbf{y}} U_{\mathbf{x}}(\mathbf{z}')] \right] P_s(\mathbf{z}') \\ &= \hat{\mathcal{I}}_{\mathbf{xy}}^t [\hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t') + \hat{\mathbf{j}}_{\mathbf{s}}(\mathbf{z}) P_{\mathbf{z}}(\mathbf{z}', t') + \mathbf{D} P_{\mathbf{s}}(\mathbf{z}) \nabla_{\mathbf{x}} P_{\mathbf{s}}(\mathbf{z})^{-1} P_{\mathbf{z}}(\mathbf{z}', t')] \\ &= \hat{\mathcal{I}}_{\mathbf{xy}}^t [\hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t') + \tilde{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}}(\mathbf{z}', t')].\end{aligned}\tag{S60}$$

This proves the correlation result. Eq. (10) in the Letter. Since we know that $\tilde{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}}(\mathbf{z}', t')$ is the initial point current at \mathbf{z} of trajectories from \mathbf{z} to \mathbf{z}' , this is a natural generalization of the second moment of the density Eq. (S59). All

final-point currents of trajectories that hit first \mathbf{y} and a time t' later \mathbf{x} , and all initial-point currents of trajectories that hit first \mathbf{x} and then \mathbf{y} contribute to the correlation of $\mathbf{J}^{\overline{U}}(\mathbf{x}, t)$ and $\rho^{\overline{U}}(\mathbf{y}, t)$.

To obtain the current correlations, we need to have a tilted generator with Fourier variables ω, ω' corresponding to $\mathbf{I}_t = \int_{\tau=0}^{\tau=t} U(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau$ and $\mathbf{I}'_t = \int_{\tau=0}^{\tau=t} V(\mathbf{x}_\tau) \circ d\mathbf{x}_\tau$, which can be by the same formalism derived as

$$\begin{aligned} \hat{L}(\mathbf{x}, \omega, \omega') &= \hat{L} - i\omega^T \cdot \hat{\mathbf{L}}_U(\mathbf{x}) - i\omega'^T \cdot \hat{\mathbf{L}}_V(\mathbf{x}) - U(\mathbf{x})^2 \omega^T \mathbf{D} \omega - V(\mathbf{x})^2 \omega'^T \mathbf{D} \omega' - 2U(\mathbf{x})V(\mathbf{x}) \omega^T \mathbf{D} \omega' \\ \hat{\mathbf{L}}_U(\mathbf{x}) &\stackrel{\text{def}}{=} U(\mathbf{x}) \hat{\mathbf{j}}_{\mathbf{x}} - \mathbf{D} \nabla_{\mathbf{x}} U(\mathbf{x}). \end{aligned} \quad (\text{S61})$$

The second moment for the scalar product of currents (trace of covariance matrix) is given by the terms that are first order in both ω and ω'

$$\begin{aligned} \left\langle \overline{\mathbf{J}^U}(\mathbf{x}, t) \cdot \overline{\mathbf{J}^U}(\mathbf{y}, t) \right\rangle_s &= \frac{1}{t^2} \langle \mathbf{I}_t \cdot \mathbf{I}'_t \rangle_s = \frac{2\text{Tr} \mathbf{D}}{t} \int d^d z U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{y}}(\mathbf{z}) P_s(\mathbf{z}) \\ &+ \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) \int d^d z \int d^d z' \left[\hat{\mathbf{L}}_{U_{\mathbf{x}}}(\mathbf{z}) \mathcal{G}(\mathbf{z}, t' | \mathbf{z}') \cdot \hat{\mathbf{L}}_{U_{\mathbf{y}}}(\mathbf{z}') P_s(\mathbf{z}') + \hat{\mathbf{L}}_{U_{\mathbf{y}}}(\mathbf{z}') \mathcal{G}(\mathbf{z}', t' | \mathbf{z}) \cdot \hat{\mathbf{L}}_{U_{\mathbf{x}}}(\mathbf{z}) P_s(\mathbf{z}) \right]. \end{aligned} \quad (\text{S62})$$

Take the first term in the last integral. Up to gradient terms ($\hat{=}$) (that vanish upon integration) we have the equalities

$$\begin{aligned} \hat{\mathbf{L}}_U(\mathbf{z}) \mathcal{G}(\mathbf{z}, t' | \mathbf{z}') \cdot \hat{\mathbf{L}}_V(\mathbf{z}') P_s(\mathbf{z}') &= U(\mathbf{z}) \hat{\mathbf{j}}_{\mathbf{z}} \mathcal{G}(\mathbf{z}, t' | \mathbf{z}') \cdot [V(\mathbf{z}') \hat{\mathbf{j}}_{\mathbf{s}}(\mathbf{z}') - P_s(\mathbf{z}') \mathbf{D} \nabla_{\mathbf{y}} P_s(\mathbf{z}')^{-1} - \mathbf{D} \nabla_{\mathbf{y}} V(\mathbf{z}')] P_s(\mathbf{z}') \\ &\hat{=} U(\mathbf{z}) V(\mathbf{z}') \hat{\mathbf{j}}_{\mathbf{z}} \cdot [\hat{\mathbf{j}}_{\mathbf{s}}(\mathbf{z}') P_s(\mathbf{z}') + P_s(\mathbf{z}') \mathbf{D} \nabla_{\mathbf{y}}] \mathcal{G}(\mathbf{z}, t' | \mathbf{z}') \\ &= U(\mathbf{z}) V(\mathbf{z}') \hat{\mathbf{j}}_{\mathbf{z}} \cdot \tilde{\mathbf{j}}_{\mathbf{z}'} P_{\mathbf{z}'}(\mathbf{z}, t). \end{aligned} \quad (\text{S63})$$

Similarly for the second term (more precisely $\mathbf{z} \leftrightarrow \mathbf{z}'$ and $U \leftrightarrow V$) we get the result

$$\left\langle \overline{\mathbf{J}^U}(\mathbf{x}, t) \cdot \overline{\mathbf{J}^U}(\mathbf{y}, t) \right\rangle_s = \frac{2\text{Tr} \mathbf{D}}{t} \int d^d z U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{y}}(\mathbf{z}) P_s(\mathbf{z}) + \hat{\mathcal{I}}_{\mathbf{x}\mathbf{y}} \left[\hat{\mathbf{j}}_{\mathbf{z}} \cdot \tilde{\mathbf{j}}_{\mathbf{z}'} P_{\mathbf{z}'}(\mathbf{z}, t) + \hat{\mathbf{j}}_{\mathbf{z}'} \cdot \tilde{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}}(\mathbf{z}', t) \right]. \quad (\text{S64})$$

This proves the variance result. Eq. (11) in the Letter. Comparing with Eqs. (S59) and (S60), this is the next generalization with contributions from currents at final and initial points from all trajectories from \mathbf{x} to \mathbf{y} and \mathbf{y} to \mathbf{x} in all times $t' \leq t$. The first extra term is purely diffusive and emerges in the overlap of $U_{\mathbf{x}}$ and $U_{\mathbf{y}}$ for $t' = t_2 - t_1 = 0$ from the non-vanishing $dW_{t_1} dW_{t_2} \neq 0$ term in the above derivation.

IV. LIMIT TO DELTA FUNCTION

We now take the limit to very small window sizes which will turn out to depend only on the properties of the two-point functions $P_{\mathbf{x}}(\mathbf{y}, t')$ for small time differences $t' = t_2 - t_1$. This allows us to derive the bounds in Eqs. (12-13) in the Letter. We consider normalized window functions such that for a window size $h \rightarrow 0$ the window function becomes a delta distribution $U_{\mathbf{x}}(\mathbf{z}) \rightarrow \delta(\mathbf{x} - \mathbf{z})$.

A. Density variance

From the second moment of the density Eq. (S59) and mean Eq. (S40) with $\text{var}_{\rho}^{\mathbf{x}}(t) \equiv \left\langle \overline{\rho^U}(\mathbf{x}, t)^2 \right\rangle_s - \left\langle \overline{\rho^U}(\mathbf{x}, t) \right\rangle_s^2$, we have

$$\text{var}_{\rho}^{\mathbf{x}}(t) = 2\hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t [P_{\mathbf{z}'}(\mathbf{z}, t') - P_s(\mathbf{z}) P_s(\mathbf{z}')]. \quad (\text{S65})$$

For window size $h \rightarrow 0$ the mean remains finite such that $\hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t [P_s(\mathbf{z}) P_s(\mathbf{z}')] \xrightarrow{h \rightarrow 0} -2P_s(\mathbf{x})^2 = O(h^0)$. Now consider

$$\hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t [P_{\mathbf{z}'}(\mathbf{z}, t')] = \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') P_{\mathbf{z}'}(\mathbf{z}, t'). \quad (\text{S66})$$

For $t' > \varepsilon > 0$, $P_{\mathbf{z}'}(\mathbf{z}, t')$ is bounded and thus $\int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') P_{\mathbf{z}'}(\mathbf{z}, t')$ is bounded by $\|P_{\mathbf{z}'}(\mathbf{z}, \varepsilon)\|_{\infty} = O(h^0)$. Contributions diverging for $h \rightarrow 0$ can thus only come from the $t' \rightarrow 0$ part of the integral, i.e. from **small time differences** $t' = t_2 - t_1$ (but **not** small absolute time t) in the Dyson series. To get the dominant divergent

contribution, we can thus set $1 - t'/t \rightarrow 1$ and replace the two-point function $P_{\mathbf{z}'}(\mathbf{z}, t')$ by the short time propagator $P_{\mathbf{y}}(\mathbf{z}, t') \rightarrow P_s(\mathbf{z}')\mathcal{G}_{\text{short}}(\mathbf{z}, t'|\mathbf{z}')$ which reads (for simplicity take $\mathbf{D} = D\mathbf{1}$, which we generalize later) [3]

$$\begin{aligned}\mathcal{G}_{\text{short}}(\mathbf{z}, t'|\mathbf{z}') &= (4\pi Dt')^{-d/2} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}' - F(\mathbf{z}')t']^2}{4Dt'} \right] \\ &= (4\pi Dt')^{-d/2} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} + \frac{2(\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{z}')t'}{4Dt'} + O(t') \right] \\ &\stackrel{\mathbf{z} \approx \mathbf{z}'}{\approx} (4\pi Dt')^{-d/2} \left[1 + \frac{1}{4D}(\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{z}') \right] \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right].\end{aligned}\quad (\text{S67})$$

We write for $t' \rightarrow 0, \mathbf{z} - \mathbf{z}' \rightarrow \mathbf{0}$,

$$\begin{aligned}\mathcal{G}_{\text{short},2} &\equiv (4\pi Dt')^{-d/2} \left[1 + \frac{1}{2D}(\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{z}') \right] \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] \\ \mathcal{G}_{\text{short},3} &\equiv (4\pi Dt')^{-d/2} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right],\end{aligned}\quad (\text{S68})$$

where $\mathcal{G}_{\text{short},2}$ can be replaced by $\mathcal{G}_{\text{short},3}$ (since $\mathbf{z} - \mathbf{z}'$ is small) if $\mathcal{G}_{\text{short},3}$ does not give zero in the integrals..
For Gaussian indicator functions

$$U_{\mathbf{x}}(\mathbf{z}) = (2\pi h^2)^{-d/2} \exp \left[-\frac{(\mathbf{z} - \mathbf{x})^2}{2h^2} \right], \quad (\text{S69})$$

we obtain for the spatial integrals

$$\begin{aligned}&\int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \mathcal{G}_{\text{short},3}(\mathbf{z}, t'|\mathbf{z}') P_s(\mathbf{z}') P_s(\mathbf{x}) \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \mathcal{G}_{\text{short},3}(\mathbf{z}, t'|\mathbf{z}') \\ &= P_s(\mathbf{x}) (2\pi h^2)^{-d} (4\pi Dt')^{-d/2} \int d^d z \int d^d z' \exp \left[-\frac{(\mathbf{z} - \mathbf{z})^2}{2h^2} - \frac{(\mathbf{z}' - \mathbf{z})^2}{2h^2} - \frac{(\mathbf{z} - \mathbf{z}')^2}{4Dt'} \right] \\ &= P_s(\mathbf{x}) (2\pi h^2)^{-d} (4\pi Dt')^{-d/2} \left(\int dx_1 \int dy_1 \exp \left[-\frac{x_1^2}{2h^2} - \frac{y_1^2}{2h^2} - \frac{(x_1 - y_1)^2}{4Dt'} \right] \right)^d \\ &= P_s(\mathbf{x}) \left[\frac{\sqrt{2Dh^2t' + h^4}}{2\sqrt{\pi}h^2\sqrt{2Dt' + h^2}\sqrt{\frac{Dt'}{h^2} + 1}} \right]^d \\ &= P_s(\mathbf{x}) (4\pi)^{-d/2} (Dt' + h^2)^{-d/2}.\end{aligned}\quad (\text{S70})$$

This implies, throughout denoting by \simeq asymptotic equality in the limit $h \rightarrow 0$ (i.e. equality of the largest order),

$$\begin{aligned}\hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')] &\simeq (4\pi)^{-d/2} \frac{P_s(\mathbf{x})}{t} \int_0^t dt' (Dt' + h^2)^{-d/2} \\ &= (4\pi)^{-d/2} \frac{P_s(\mathbf{x})}{t} \times \begin{cases} -\frac{h^{2-d}}{D(1-\frac{d}{2})} + \frac{(D+h^2)^{1-\frac{d}{2}}}{D(1-\frac{d}{2})} & \text{for } d \neq 2 \\ -\frac{\log(h^2)}{D} + \frac{\log(D+h^2)}{D} & \text{otherwise} \end{cases} \\ &\simeq (4\pi)^{-d/2} \frac{2P_s(\mathbf{x})}{Dt} \times \begin{cases} \frac{h^{2-d}}{d-2} & \text{for } d > 2 \\ -\log(h) & \text{for } d = 2 \end{cases}.\end{aligned}\quad (\text{S71})$$

This gives for Gaussian U with width h the result

$$\text{var}_{\rho}^{\mathbf{x}}(t) \stackrel{h \rightarrow 0}{\simeq} (4\pi)^{-d/2} \frac{4P_s(\mathbf{x})}{Dt} \times \begin{cases} \frac{h^{2-d}}{d-2} & \text{for } d > 2 \\ -\log(h) & \text{for } d = 2 \end{cases}, \quad (\text{S72})$$

where only the numerical prefactor changes if we choose other indicator functions, since the relevant part (close to \mathbf{x}) of any finite size window function can be bounded from above and below by Gaussian window functions.

If \mathbf{D} is not isotropic we transform to coordinates where the diffusion matrix is diagonal, $\mathbf{D} = \text{diag}(D_1, \dots, D_d)$ where we then need to integrate

$$\int_0^t dt' \prod_{i=1}^d (D_i t' + h^2)^{-1/2}, \quad (\text{S73})$$

which can be bounded by

$$(\max_j (D_j) t' + h^2)^{-1/2} \leq (D_i t' + h^2)^{-1/2} \leq (\min_j (D_j) t' + h^2)^{-1/2}, \quad (\text{S74})$$

implying that in the final result D can be replaced by $\tilde{D} \in [\min(D_i), \max(D_i)]$,

$$\text{var}_\rho^{\mathbf{x}}(t) \stackrel{h \rightarrow 0}{\simeq} (4\pi)^{-d/2} \frac{4P_s(\mathbf{x})}{\tilde{D}t} \times \begin{cases} \frac{h^{2-d}}{d-2} & \text{for } d > 2 \\ -\log(h) & \text{for } d = 2 \end{cases}. \quad (\text{S75})$$

The entries D_i of the diagonalized \mathbf{D} are the eigenvalues, hence in general $\tilde{D} \in [\lambda_{\min}, \lambda_{\max}]$ is bounded by the lowest and highest eigenvalues λ_{\min} and λ_{\max} of the matrix \mathbf{D} .

B. Correlation of current and density

Now consider the small-window limit for correlations with $\mathbf{x} = \mathbf{y}$ defined as $\mathbf{C}_{\mathbf{J}_\rho}^{\mathbf{x}\mathbf{x}}(t) \equiv \langle \overline{\mathbf{J}^U}(\mathbf{x}, t) \overline{\rho^U}(\mathbf{y}, t) \rangle_s - \langle \overline{\mathbf{J}^U}(\mathbf{x}, t) \rangle_s \langle \overline{\rho^U}(\mathbf{y}, t) \rangle_s$, given according to Eq. (S60) by

$$\mathbf{C}_{\mathbf{J}_\rho}^{\mathbf{x}\mathbf{x}}(t) = \hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t \left[\hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t') + \tilde{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}', t') - 2\mathbf{j}_s(\mathbf{z}) P_s(\mathbf{z}') \right]. \quad (\text{S76})$$

Recall that $\hat{\mathbf{j}}_{\mathbf{z}} = \mathbf{F}(\mathbf{z}) - \mathbf{D}\nabla_{\mathbf{z}}$. As always the term involving the mean values is finite for $h \rightarrow 0$. We first look at $\hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t[\hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t')]$, again first for isotropic diffusion $\mathbf{D} = D\mathbf{1}$.

Here we have to use $\mathcal{G}_{\text{short},2}$, see Eq. (S68), since the integrals over $\mathcal{G}_{\text{short},3}$ vanish. Hence consider

$$\begin{aligned} \hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t[\hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t')] &= \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t} \right) \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t') \\ &\simeq \frac{P_s(\mathbf{x})}{t} \int_0^t dt' \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') [\mathbf{F}(\mathbf{z}) - D\nabla_{\mathbf{z}}] \mathcal{G}_{\text{short},2}(\mathbf{z}, t'|\mathbf{z}'), \end{aligned} \quad (\text{S77})$$

where we can use $\hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t[\mathbf{F}(\mathbf{z}) P_{\mathbf{z}'}(\mathbf{z}, t')] \simeq \mathbf{F}(\mathbf{x}) \hat{\mathcal{I}}_{\mathbf{x}\mathbf{x}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')] \simeq \mathbf{F}(\mathbf{x}) \times$ (S71) and we compute

$$\begin{aligned} &\nabla_{\mathbf{z}} \mathcal{G}_{\text{short},2}(\mathbf{z}, t'|\mathbf{z}') \\ &= (4\pi Dt')^{-d/2} \left[1 + \frac{1}{2D} (\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{z}') \right] \nabla_{\mathbf{z}} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] + (4\pi Dt')^{-d/2} \frac{\mathbf{F}(\mathbf{z}')}{2D} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] \\ &= -(4\pi Dt')^{-d/2} \left[1 + \frac{1}{2D} (\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{z}') \right] \frac{\mathbf{z} - \mathbf{z}'}{2Dt'} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] + (4\pi Dt')^{-d/2} \frac{\mathbf{F}(\mathbf{z}')}{2D} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right]. \end{aligned} \quad (\text{S78})$$

By symmetry, the spatial integrals over $(\mathbf{z} - \mathbf{z}') \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right]$ vanish and we are left to compute

$$\begin{aligned} &-D \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \nabla_{\mathbf{z}} \mathcal{G}_{\text{short},2}(\mathbf{z}, t'|\mathbf{z}') \\ &\simeq -D(4\pi Dt')^{-d/2} \int d^d x \int d^d y U(\mathbf{x}) U(\mathbf{y}) \left(-\frac{[(\mathbf{x} - \mathbf{y}) \cdot \mathbf{F}(\mathbf{y})](\mathbf{x} - \mathbf{y})}{4D^2 t'} + \frac{\mathbf{F}(\mathbf{y}')}{2D} \right) \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right], \end{aligned} \quad (\text{S79})$$

where the second term gives $-\frac{1}{2}\mathbf{F}(\mathbf{x})\hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')]$. The remaining term, noting that $\mathbf{F}(\mathbf{z}') \simeq \mathbf{F}(\mathbf{x})$ and integrating out all directions except k for the $F_k(\mathbf{x})$ component (by symmetry $(z_i - z'_i)(z_j - z'_j)$ integrate to zero if $i \neq j$), becomes

$$\begin{aligned} & \frac{(4\pi Dt')^{-d/2}}{4Dt'} \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') [(\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{x})] (\mathbf{z} - \mathbf{z}') \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] \\ &= \frac{\mathbf{F}(\mathbf{x})}{4Dt'} \int dz_1 \int dz'_1 U^1(z_1) U^1(z'_1) (z_1 - z'_1)^2 \mathcal{G}_{\text{short}, 3, \text{one-dim}}(z_1, t' | z'_1) \\ &= \frac{\mathbf{F}(\mathbf{x})}{4Dt'} \frac{Dh^2 t'}{\sqrt{\pi} (Dt' + h^2)^{\frac{3}{2}}} \\ &= \frac{\mathbf{F}(\mathbf{x}) h^2}{4\sqrt{\pi} (Dt' + h^2)^{\frac{3}{2}}}. \end{aligned} \quad (\text{S80})$$

This term is subdominant as we see from the time integral

$$\begin{aligned} \hat{\mathcal{I}}_{\mathbf{xx}}^t[-D\nabla_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t')] &\simeq -\frac{\mathbf{F}(\mathbf{x})}{2} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')] + \frac{P_s(\mathbf{x}) \mathbf{F}(\mathbf{x}) h^2}{4\sqrt{\pi} t} \underbrace{\int_0^t dt' (Dt' + h^2)^{-\frac{3}{2}}}_{h^{-1}} \\ &\simeq -\frac{\mathbf{F}(\mathbf{x})}{2} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')]. \end{aligned} \quad (\text{S81})$$

Hence, overall we get

$$\hat{\mathcal{I}}_{\mathbf{xx}}^t[\hat{\mathbf{j}}_{\mathbf{z}} P_{\mathbf{z}'}(\mathbf{z}, t')] = \frac{\mathbf{F}(\mathbf{x})}{2} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')]. \quad (\text{S82})$$

The generalization to non-isotropic \mathbf{D} only changes $\hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}'}(\mathbf{z}, t')]$ but Eq. (S82) is retained.

Now consider $\hat{\mathcal{I}}_{\mathbf{xx}}^t[\hat{\mathbf{j}}_{\mathbf{z}}^\dagger P_{\mathbf{z}}(\mathbf{z}', t')]$. Since this involves derivatives of both \mathcal{G} and P_s (at the initial point) we instead take the form $\hat{\mathbf{j}}_{\mathbf{z}}^\dagger = \hat{\mathbf{j}}_s(\mathbf{z}) - \hat{\mathbf{j}}_g(\mathbf{z}) = \mathbf{j}_s(\mathbf{z})/P_s(\mathbf{z}) + DP_s(\mathbf{z})\nabla_{\mathbf{z}} P_s(\mathbf{z})^{-1}$ such that $\hat{\mathbf{j}}_{\mathbf{z}}^\dagger P_{\mathbf{z}}(\mathbf{z}', t') = [\mathbf{j}_s(\mathbf{z}) + DP_s(\mathbf{z})\nabla_{\mathbf{z}}]\mathcal{G}(\mathbf{z}', t' | \mathbf{z})$ giving

$$\begin{aligned} \hat{\mathcal{I}}_{\mathbf{xx}}^t[\hat{\mathbf{j}}_{\mathbf{z}}^\dagger P_{\mathbf{z}}(\mathbf{z}', t')] &= \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t}\right) \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') [\mathbf{j}_s(\mathbf{z}) + DP_s(\mathbf{z})\nabla_{\mathbf{z}}] \mathcal{G}(\mathbf{z}', t' | \mathbf{z}) \\ &\simeq \frac{\mathbf{j}_s(\mathbf{x})}{P_s(\mathbf{x})} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}}(\mathbf{z}', t')] + DP_s(\mathbf{x}) \int_0^t dt' \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \nabla_{\mathbf{z}} \mathcal{G}_{\text{short}, 2}(\mathbf{z}', t' | \mathbf{z}), \end{aligned} \quad (\text{S83})$$

where (note that $\mathcal{G}_{\text{short}, 3}(\mathbf{z}, t' | \mathbf{z}') = \mathcal{G}_{\text{short}, 3}(\mathbf{z}', t' | \mathbf{z})$)

$$\begin{aligned} & \nabla_{\mathbf{z}} \mathcal{G}_{\text{short}, 2}(\mathbf{z}', t' | \mathbf{z}) \\ &= (4\pi Dt')^{-d/2} \left[1 + \frac{1}{2D} (\mathbf{z}' - \mathbf{z}) \cdot \mathbf{F}(\mathbf{z}') \right] \nabla_{\mathbf{z}} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] + (4\pi Dt')^{-d/2} \frac{-\mathbf{F}(\mathbf{z})}{2D} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] \\ &= -(4\pi Dt')^{-d/2} \left[1 + \frac{1}{2D} (\mathbf{z} - \mathbf{z}') \cdot \mathbf{F}(\mathbf{z}') \right] \frac{\mathbf{z} - \mathbf{z}'}{2Dt'} \exp \left[-\frac{[\mathbf{z} - \mathbf{z}']^2}{4Dt'} \right] - \frac{\mathbf{F}(\mathbf{z}')}{2D} \mathcal{G}_{\text{short}, 3}(\mathbf{z}', t' | \mathbf{z}). \end{aligned} \quad (\text{S84})$$

As before, asymptotically only the last term contributes, giving

$$\begin{aligned} \hat{\mathcal{I}}_{\mathbf{xx}}^t[\hat{\mathbf{j}}_{\mathbf{z}}^\dagger P_{\mathbf{z}}(\mathbf{z}', t')] &\simeq \frac{\mathbf{j}_s(\mathbf{x})}{P_s(\mathbf{x})} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}}(\mathbf{z}', t')] + DP_s(\mathbf{x}) \int_0^t dt' \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \nabla_{\mathbf{z}} \mathcal{G}_{\text{short}, 2}(\mathbf{z}', t' | \mathbf{z}) \\ &\simeq \frac{\mathbf{j}_s(\mathbf{x})}{P_s(\mathbf{x})} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}}(\mathbf{z}', t')] + DP_s(\mathbf{x}) \int_0^t dt' \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \frac{-\mathbf{F}(\mathbf{z}')}{2D} \mathcal{G}_{\text{short}, 3}(\mathbf{z}', t' | \mathbf{z}) \\ &\simeq \left[\frac{\mathbf{j}_s(\mathbf{x})}{P_s(\mathbf{x})} - \frac{\mathbf{F}(\mathbf{x})}{2} \right] \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}}(\mathbf{z}', t')]. \end{aligned} \quad (\text{S85})$$

Overall, this gives for the correlation result (same form for non-isotropic diffusion)

$$\mathbf{C}_{\mathbf{J}_\rho}^{\mathbf{xx}}(t) \xrightarrow{h \rightarrow 0} \frac{\mathbf{j}_s(\mathbf{x})}{P_s(\mathbf{x})} \hat{\mathcal{I}}_{\mathbf{xx}}^t[P_{\mathbf{z}}(\mathbf{z}', t')] \simeq \frac{\mathbf{j}_s(\mathbf{x})}{2P_s(\mathbf{x})} \text{var}_{\rho}^{\mathbf{x}}(t). \quad (\text{S86})$$

Eqs. (S83), (S85) and (S86) prove the equations between Eq. (12) and (13) in the Letter.

C. Current variance

We now turn to the current variance result, see Eq. (S64) for $\mathbf{x} = \mathbf{y}$,

$$\text{var}_{\mathbf{J}}^{\mathbf{x}}(t) = \frac{2\text{Tr}\mathbf{D}}{t} \int d^d z U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}) P_{\mathbf{s}}(\mathbf{z}) + 2\hat{\mathcal{I}}_{\mathbf{xx}}^t \left[\hat{\mathbf{j}}_{\mathbf{z}} \cdot \tilde{\mathbf{j}}_{\mathbf{z}'} P_{\mathbf{z}'}(\mathbf{z}, t') - \mathbf{j}_{\mathbf{s}}(\mathbf{z}) \cdot \mathbf{j}_{\mathbf{s}}(\mathbf{z}') \right]. \quad (\text{S87})$$

The first term for $h \rightarrow 0$ gives

$$\frac{2\text{Tr}\mathbf{D}}{t} \int d^d x U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}) P_{\mathbf{s}}(\mathbf{z}) \simeq \frac{2\text{Tr}\mathbf{D}}{t} P_{\mathbf{s}}(\mathbf{x}) U_{\mathbf{x}}^h(\mathbf{x}), \quad (\text{S88})$$

where $U_{\mathbf{x}}^h(\mathbf{x}) \propto h^{-d}$ is the height of the delta-function approximation, e.g. $U_{\mathbf{x}}^h(\mathbf{x}) = (2\pi)^{-d/2} h^{-d}$ for Gaussian $U_{\mathbf{x}}$. In the derivation above this term occurred from cross correlations $dW_{t_1} dW_{t_1} = dt' \neq 0$ in the noise part, hence it can be seen to come from zero time-differences $t' = t_2 - t_1 = 0$. Such a term does not appear in the density variance or density-current correlation since there $dt_1 dt_2 = 0$ and $dt_1 dW_{t_2} = 0$ would occur instead of $dW_{t_1} dW_{t_2}$.

Due to the fast h^{-d} divergence, the dominant limit does not depend on terms with no or only one derivative since they were shown to scale at most as h^{2-d} . The only new term is the second derivative for which we see that

$$\begin{aligned} & \int_0^t dt' D \nabla_{\mathbf{z}} \cdot (-\nabla_{\mathbf{z}'}) \mathcal{G}_{\text{short},3}(\mathbf{z}, t' | \mathbf{z}') \\ &= \int_0^t D \nabla_{\mathbf{z}}^2 \mathcal{G}_{\text{short},3}(\mathbf{z}, t' | \mathbf{z}') \\ &= \int_0^t dt' \partial_{t'} \mathcal{G}_{\text{short},3}(\mathbf{z}, t' | \mathbf{z}') \\ &= [\mathcal{G}_{\text{short},3}(\mathbf{z}, t' | \mathbf{z}')]_0^t \\ &= \mathcal{G}_{\text{short},3}(\mathbf{z}, t | \mathbf{z}') - \delta(\mathbf{z} - \mathbf{z}'), \end{aligned} \quad (\text{S89})$$

such that

$$\begin{aligned} \hat{\mathcal{I}}_{\mathbf{xx}}^t \left[\hat{\mathbf{j}}_{\mathbf{z}} \cdot \tilde{\mathbf{j}}_{\mathbf{z}'} P_{\mathbf{z}'}(\mathbf{z}, t') \right] &\simeq -D^2 \frac{P_{\mathbf{s}}(\mathbf{x})}{t} \int_0^t dt' \int d^d z \int d^d z' U_{\mathbf{x}}^h(\mathbf{z}) U^h(\mathbf{z}) \nabla_{\mathbf{z}} \cdot \nabla_{\mathbf{z}'} \mathcal{G}_{\text{short},3}(\mathbf{z}, t' | \mathbf{z}') \\ &\simeq -D \frac{P_{\mathbf{s}}(\mathbf{x})}{t} \int d^d z \int d^d z' U_{\mathbf{x}}^h(\mathbf{z}) U^h(\mathbf{z}') \delta(\mathbf{z} - \mathbf{z}') \\ &\simeq -D \frac{P_{\mathbf{s}}(\mathbf{x})}{t} U^h(\mathbf{x}). \end{aligned} \quad (\text{S90})$$

For non-isotropic $\mathbf{D} \neq D\mathbf{1}$, we move the basis where \mathbf{D} is diagonal where

$$D^2 \nabla_{\mathbf{z}} \cdot \nabla_{\mathbf{z}'} \longrightarrow \sum_{i=1}^d D_i^2 \partial_{z_i} \partial_{z_{i'}}. \quad (\text{S91})$$

The operator we need is $\nabla_{\mathbf{z}} \mathbf{D} \nabla_{\mathbf{z}'} = \sum_i D_i \partial_{z_i} \partial_{z_{i'}}$ so we bound one of the D_i in D_i^2 by $D' \in [\min(D_i), \max(D_i)]$ such that we get

$$\hat{\mathcal{I}}_{\mathbf{xx}}^t \left[\hat{\mathbf{j}}_{\mathbf{z}} \cdot \tilde{\mathbf{j}}_{\mathbf{z}'} P_{\mathbf{z}'}(\mathbf{z}, t') \right] \simeq -D' \frac{P_{\mathbf{s}}(\mathbf{x})}{t} U^h(\mathbf{x}). \quad (\text{S92})$$

Since $\text{Tr}\mathbf{D} = \sum_i D_i$ we have $\tilde{D}' \equiv \frac{\text{Tr}\mathbf{D} - D'}{d-1} \in [\min(D_i), \max(D_i)]$ and we can write

$$\text{var}_{\mathbf{J}}^{\mathbf{x}}(t) \simeq \frac{2\text{Tr}\mathbf{D}}{t} P_{\mathbf{s}}(\mathbf{x}) U^h(\mathbf{x}) - 2D' \frac{P_{\mathbf{s}}(\mathbf{x})}{t} U^h(\mathbf{x}) = \frac{2\tilde{D}'}{t} P_{\mathbf{s}}(\mathbf{x}) (d-1) U^h(\mathbf{x}), \quad (\text{S93})$$

where $U^h(\mathbf{x}) \propto h^{-d}$. This proves Eq. (13) in the Letter. Thus, we see that the current fluctuations diverge for $h \rightarrow 0$, except in one-dimensional space where $d-1=0$.

D. Limit to delta function in the one-dimensional case

In the one-dimensional case, the variance of empirical density and current remain finite for $h \rightarrow 0$ which allows to take the limit to $U_x(x') = \delta(x - x')$. In terms of the stochastic integrals, the one-dimensional case is much simpler, since any one-dimensional function $U_x(x')$ possesses an antiderivative – a primitive function $\mathcal{U}_x(x') = \int^{x'} U_x(x'') dx''$ such that $U_x(x') = \partial_{x'} \mathcal{U}_x(x')$. This implies for the Stratonovich integral that

$$t\overline{J^U}(x, t) = \int_0^t U_x(x_\tau) \circ dx_\tau = \mathcal{U}_x(x_t) - \mathcal{U}_x(x_0). \quad (\text{S94})$$

Thus, the stochastic current is no longer a functional but only a function of the initial- and end-point of the trajectory. Its moments are directly accessible, e.g.

$$\left\langle \overline{J^U}(x, t) \right\rangle_s = \frac{1}{t^2} \left\langle [\mathcal{U}_x(x_t) - \mathcal{U}_x(x_0)]^2 \right\rangle_s = \frac{1}{t^2} \int dz \int dz' [\mathcal{U}_x(z) - \mathcal{U}_x(z')]^2 P_{z'}(z, t). \quad (\text{S95})$$

If U is Gaussian, then \mathcal{U}_x is the error function such that $[\mathcal{U}_x(x) - \mathcal{U}_x(y)]^2 \leq 1$ and thus $\left\langle \overline{J^U}(x, t) \right\rangle_s \leq 1/t^2$. This also holds in the limit of a delta function where the primitive function becomes a Heaviside step function and we get that the current can only be 0 or $\pm t^{-2}$, see Fig. S1. The current defined with a delta-function at x simply counts the net number of crossing through x such that all crossings except maybe one cancel out. To obtain a $1/t$ -term as in large deviations one would need to have a steady-state current which could e.g. be achieved by generalizing to periodic boundary conditions. Then the current would depend on the initial and final point and, in addition, also on the net number of crossings of the full interval between the boundaries of the system.

Fig. S1 shows the time-integrated density and current, i.e. the empirical density and current Eq. (S1) multiplied by the total time t . Fluctuations remain in the same order of magnitude for $h \rightarrow 0$ (see Fig. S1c,e). We see that the time-integrated current is bounded by 1 which is due to the fact that it simply counts the net number of crossings. According to Eq. (S94) it only depends on the initial-point x_0 and end-point x_t , in this case x_{10} .

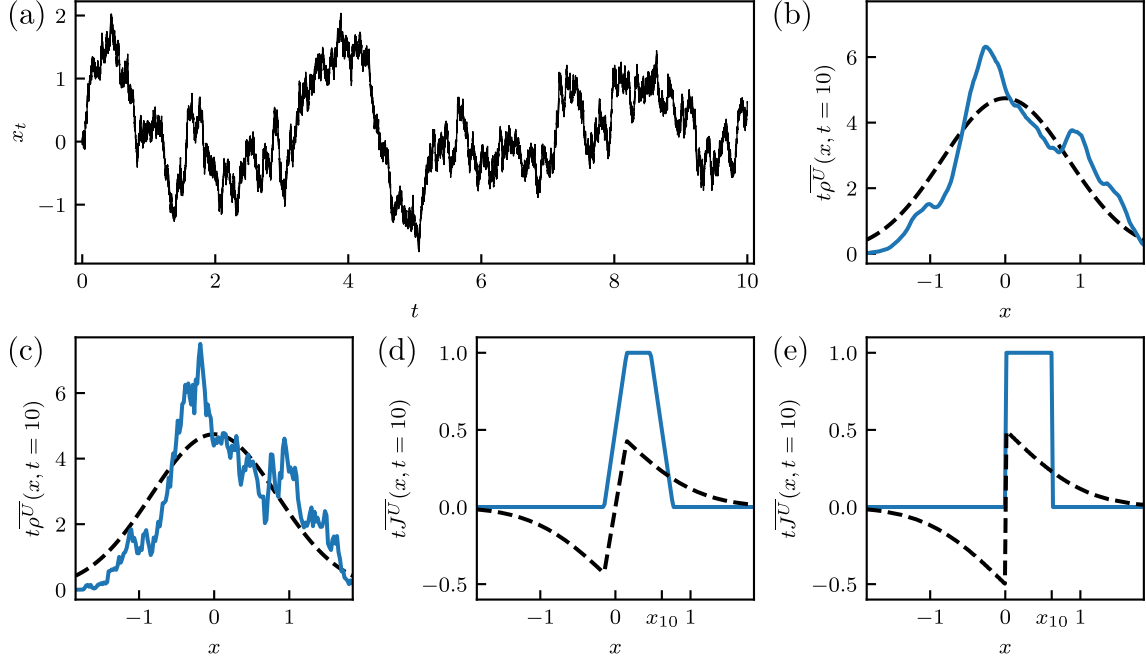


FIG. S1. **(a)** One-dimensional Brownian motion in a harmonic potential (see Eq. (S99) in 1d with $r = \sqrt{2}$ and $D = 1$) starting at $x_0 = 0$ and ending at $x_{10} = 0.62$. **(b)** Time-integrated density of the trajectory in (a) as a function of x for normalized window function $U_x(x') = h^{-1} \mathbf{1}_{|x-x'| \leq h/2}$ with width $h = 0.3$. The dashed line shows the expectation value of the time-integrated density conditioned on $\mathbf{x}_0 = 0$. **(c)** As in (b) with width $h = 0.001$. **(d)** Time-integrated current for window as in (b) width $h = 0.3$. The dashed line shows the expectation value of the time-integrated current conditioned on $\mathbf{x}_0 = 0$. **(e)** As in (d) for width $h = 0.001$.

V. CONNECTION TO LARGE DEVIATION THEORY

Recall the notation (Eq. (6) in the Letter)

$$\hat{I}_{\mathbf{x}\mathbf{y}}^t[\cdot] = \frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t}\right) \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{y}}(\mathbf{z}') [\cdot]. \quad (\text{S96})$$

We have $\int_{t'}^\infty dt'' [P_{\mathbf{y}}(\mathbf{x}, t'') - P_{\mathbf{s}}(\mathbf{x})] \rightarrow 0$ for $t' \rightarrow \infty$ since $P_{\mathbf{y}}(\mathbf{x}, t') \xrightarrow{t' \rightarrow \infty} P_{\mathbf{s}}(\mathbf{x})$ and $\hat{\mathbf{j}}_{\mathbf{x}} P_{\mathbf{y}}(\mathbf{x}, t') \xrightarrow{t' \rightarrow \infty} \mathbf{j}_{\mathbf{s}}(\mathbf{x})$ with exponentially decaying deviations. This implies that for large t , we can replace $\frac{1}{t} \int_0^t dt' \left(1 - \frac{t'}{t}\right)$ by $\frac{1}{t} \int_0^\infty dt'$ in the results Eqs. (7) and (11) in the Letter (i.e. Eqs. (S59) and (S64) after we subtract the squared mean). Thus, we see that the variance results all scale as $1/t$ for large t . This replacement of integrals and the scaling are confirmed by a spectral expansion [8] and the $1/t$ -scaling for large times is expected from the central limit theorem and large deviation theory.

According to the central limit theorem, the probability distributions $p(A_t = a)$ for $A_t = \overline{\rho^U}(\mathbf{x}, t)$ and $A_t = \overline{\mathbf{J}^U}(\mathbf{x}, t)$ become Gaussian for large t such that we obtain the large deviation rate function

$$I(a) = -\lim_{t \rightarrow \infty} \frac{1}{t} \ln p(A_t = a) = \frac{(a - \mu)^2}{2\sigma_A^2}, \quad (\text{S97})$$

where the mean μ is given by Eq. (S40) or Eq. (S41) and the large deviation variance σ_A^2 follows by the above arguments from Eqs. (S59) and (S64) for $\mathbf{x} = \mathbf{y}$ as

$$\begin{aligned} \sigma_{\rho^U}^2 &\equiv \lim_{t \rightarrow \infty} t \text{var}_{\rho}^{\mathbf{x}}(t) = 2 \int_0^\infty dt' \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') [P_{\mathbf{z}'}(\mathbf{z}, t) - P_{\mathbf{s}}(\mathbf{z}) P_{\mathbf{s}}(\mathbf{z}')] \\ \sigma_{\mathbf{J}^U}^2 &\equiv \lim_{t \rightarrow \infty} t \text{var}_{\mathbf{J}}^{\mathbf{x}}(t) = 2 \text{Tr} \mathbf{D} \int d^d z U_{\mathbf{x}}^2(\mathbf{z}) P_{\mathbf{s}}(\mathbf{z}) + 2 \int_0^\infty dt' \int d^d z \int d^d z' U_{\mathbf{x}}(\mathbf{z}) U_{\mathbf{x}}(\mathbf{z}') \left[\hat{\mathbf{j}}_{\mathbf{z}} \cdot \tilde{\mathbf{j}}_{\mathbf{z}'} P_{\mathbf{z}'}(\mathbf{z}, t) - \mathbf{j}_{\mathbf{s}}(\mathbf{z}) \mathbf{j}_{\mathbf{s}}(\mathbf{z}') \right]. \end{aligned} \quad (\text{S98})$$

For any Lebesgue integrable window function U (i.e. if the window size h fulfills $h > 0$), and in $d = 1$ even for the delta-function, this variance is finite and the central limit theorem applies as described above. The parabolic rate function for a two dimensional system with finite window size $h > 0$ is shown for the density $\overline{\rho^U}(\mathbf{x}, t)$ and current $\overline{\mathbf{J}^U}(\mathbf{x}, t)$ in Fig. S2. The agreement of the simulation and the predicted variance Eq. (S98) is well confirmed.

If we instead take $h \rightarrow 0$ or $h = 0$ in multi-dimensional space $d \geq 2$, the variances diverges (see Eqs. (S75) and (S93)). Thus, the central limit theorem no longer applies, allowing for non-Gaussian statistics [11]. Fig. S3 depicts the distribution of the empirical density $\rho^U(\mathbf{x}, t)$ in a fixed point \mathbf{x} for different times t and window sizes h . We see that the distribution becomes non-Gaussian for small h , in particular the most probable value departs from the mean and approaches 0. Even though a Gaussian distribution is restored for longer times (see Fig. S3b), for even smaller window sizes the distribution again becomes non-Gaussian (see Fig. S3c).

For $h = 0$, $\overline{\rho^U}(\mathbf{x}, t) = 0$ is the most probable value for all t , whereas $\langle \overline{\rho^U}(\mathbf{x}, t) \rangle_{\mathbf{s}} \neq 0$, implying non-Gaussian statistics. In the language of large deviations, this means that the rate function is not parabolic and in particular that the probability density does not concentrate around the mean. This behavior can be understood in terms of the recurrence time, i.e. the time it takes to revisit the window covered by U . Since the probability of Brownian motion to hit a certain point in d -dimensional space is exactly zero for $d \geq 2$, the recurrence time diverges as $h \rightarrow 0$ (i.e. $U_{\mathbf{x}}(\mathbf{x}') \rightarrow \delta(\mathbf{x} - \mathbf{x}')$ for $d \geq 2$). To apply the central limit theorem, many visits of the window are required, which is for $h = 0$ not possible, not even for arbitrarily large t .

However, even in the limit $h \rightarrow 0$ (i.e. $U_{\mathbf{x}}(\mathbf{x}') \rightarrow \delta(\mathbf{x} - \mathbf{x}')$), the probability distributions of $\overline{\rho^U}(\mathbf{x}, t)$ and $\overline{\mathbf{J}^U}(\mathbf{x}, t)$ are still normalized and have finite means $P_{\mathbf{s}}(\mathbf{x})$ and $\mathbf{j}_{\mathbf{s}}(\mathbf{x})$. The divergent variance implies power-law tails with exponent $\alpha \leq 3$ [11]. The normalization implies $\alpha > 1$ and since the empirical density is non-negative $\overline{\rho^U}(\mathbf{x}, t) \geq 0$ the finite mean implies $\alpha > 2$.

For $h = 0$ in $d \geq 2$ where the central limit theorem does not apply, the rate function $I(a) = -\lim_{t \rightarrow \infty} \frac{1}{t} \ln p(A_t = a)$, could be everywhere 0 or ∞ since if $\ln p(A_t = a)$ does not become linear in t , i.e. $p(A_t = a)$ is sub- or super-exponential in t [12]. If $p(A_t = a)$ is exponential in time, then the power-law tails imply that $I(a)$ is logarithmic and therefore not convex. In any of those cases, power law tails are not compatible with strictly convex rate functions and therefore a rate function for an empirical density or current with $U_{\mathbf{x}}(\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ cannot be obtained from the Gärtner-Ellis theorem (since then it would be strictly convex [12]). This implies the necessity to modify some existing interpretations of large deviations of empirical currents and densities in continuous multidimensional space [2, 13, 14].

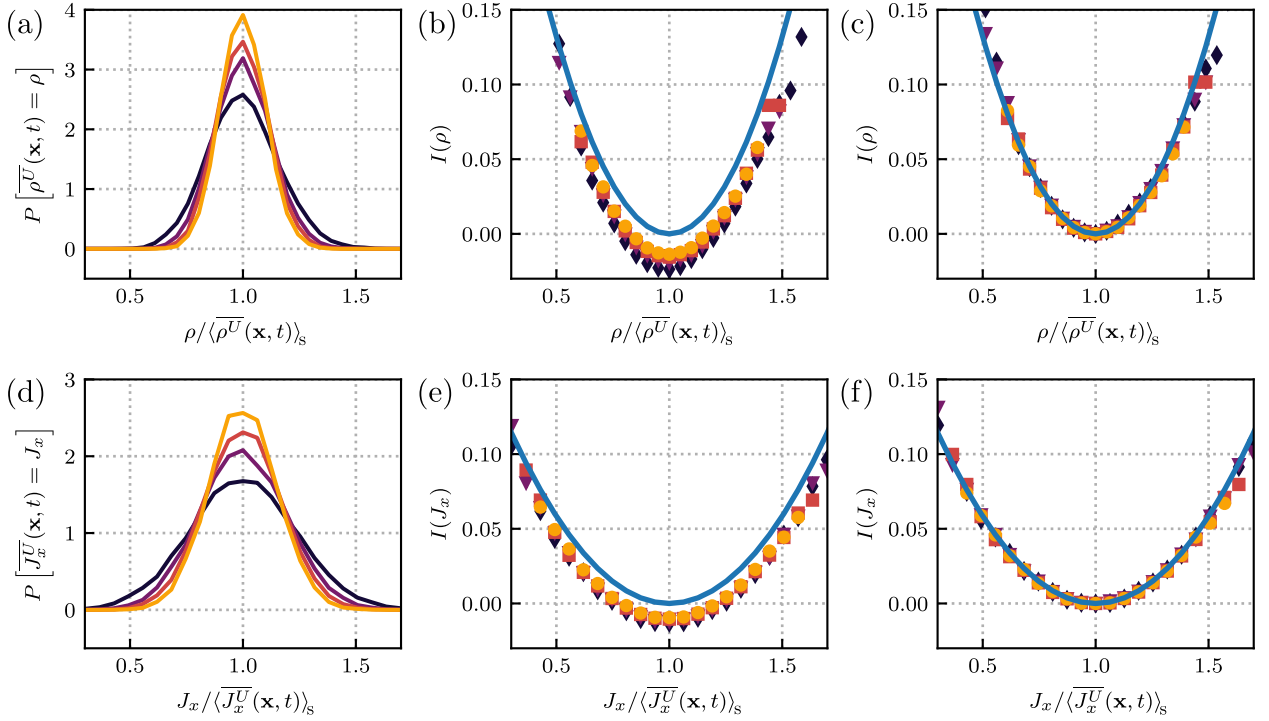


FIG. S2. **(a)** Simulation of the probability density function of the empirical density $\bar{\rho}^U(\mathbf{x}, t)$ for $\mathbf{x} = (0, 1)$ and Gaussian window function $U_{\mathbf{x}}(\mathbf{x}')$ with width $h = 0.5$. The underlying process is the two-dimensional driven Ornstein-Uhlenbeck process Eq. (S99) with $r = D = 1$ and $\Omega = 3$ with \mathbf{x}_0 from the steady state. Colors of lines and symbols throughout denote $t = 40, 60, 80, 100$ from dark to bright. The simulated probability density were obtained from histograms of 2×10^4 trajectories for each set of parameters. **(b)** Parabolic rate function with variance from Eq. (S98) (line) and simulated rate function $-\frac{1}{t} \ln P[\bar{\rho}^U(\mathbf{x}, t) = \rho]$ for $t = 40, 60, 80, 100$ (symbols). **(c)** Curves from (b) shifted to zero at the mean $\rho = \langle \bar{\rho}^U(\mathbf{x}, t) \rangle_s$. **(d-f)** As in (a-c) but for the current $\bar{\mathbf{J}}^U$ instead of the density $\bar{\rho}^U$.

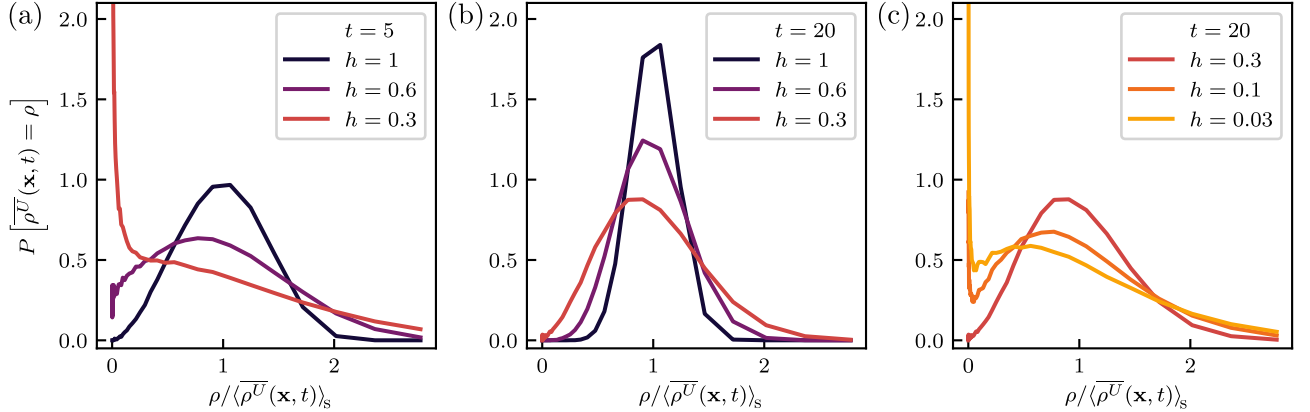


FIG. S3. Simulation of the probability density function of the empirical density $\bar{\rho}^U(\mathbf{x}, t)$ for $\mathbf{x} = (0, 1)$ and Gaussian window function $U_{\mathbf{x}}(\mathbf{x}')$ with width h for the two-dimensional driven Ornstein-Uhlenbeck process Eq. (S99) with $r = D = 1$ and $\Omega = 3$ with \mathbf{x}_0 from the steady state. The simulated probability density were obtained from histograms of 2×10^5 trajectories for each set of parameters.

VI. NUMERICAL AND ANALYTICAL EVALUATION USED FOR THE FIGURES

This section gives further parameters and all details necessary to reproduce all figures in the Letter and the SM.

A. Analytical results for the two-dimensional Ornstein-Uhlenbeck process

For the two-dimensional numerical and analytical results shown in Fig. 2 and 3 in the Letter and Fig. S2 and S3, we use the two-dimensional Ornstein-Uhlenbeck process given by the Langevin equation

$$d\mathbf{x}_t = \mathbf{F}_{\text{rot}}(\mathbf{x}_t)dt + \sqrt{2D}d\mathbf{W}_t, \quad (\text{S99})$$

with drift field $\mathbf{F}_{\text{rot}}(\mathbf{x}) = -\Theta\mathbf{x}$ where $\Theta = \begin{bmatrix} r & -\Omega \\ \Omega & r \end{bmatrix}$, $r > 0$. The drift part splits into

$$\mathbf{F}_{\text{rot}}(\mathbf{x}) = -D\{\nabla\phi(\mathbf{x})\} + \mathbf{j}_s(\mathbf{x})/P_s(\mathbf{x}), \quad (\text{S100})$$

with potential $\phi(\mathbf{x}) = \frac{r}{2D}\mathbf{x}^T\mathbf{x}$ and steady-state density and current

$$P_s(\mathbf{x}) = \frac{r}{2\pi D}e^{-r\frac{\mathbf{x}^T\mathbf{x}}{2D}} \\ \mathbf{j}_s(\mathbf{x}) = (\mathbf{F} - D\nabla_{\mathbf{x}})P_s(\mathbf{x}) = \Omega P_s(\mathbf{x}) \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}. \quad (\text{S101})$$

A straightforward left-right decomposition [7] gives the propagator/two-point function

$$\mathcal{G}(\mathbf{x}, t'|\mathbf{x}_0) = \frac{r}{2\pi D(1 - e^{-2rt'})} \exp \left[\frac{-r \left(\mathbf{x} - e^{-rt'} \begin{bmatrix} \cos(\Omega t') & \sin(\Omega t') \\ -\sin(\Omega t') & \cos(\Omega t') \end{bmatrix} \mathbf{x}_0 \right)^2}{2D(1 - e^{-2rt'})} \right] \\ P(\mathbf{x}, t'; \mathbf{x}_0, 0) = \mathcal{G}(\mathbf{x}, t'|\mathbf{x}_0)P_s(\mathbf{x}_0). \quad (\text{S102})$$

We then analytically solve the necessary Gaussian integrals for Gaussian window functions

$$U_{\mathbf{x}}(\mathbf{z}) = (2\pi h^2)^{-d/2} \exp \left[-\frac{(\mathbf{z} - \mathbf{x})^2}{2h^2} \right], \quad (\text{S103})$$

and numerically solve the remaining t' -integral in Eq. (S96). This enables a very fast and stable (even for very small coarse graining where numerical spatial integrals would eventually fail) computation of the second moments shown in Figures 2 and 3. The analytical integrals were performed with the Python-based computer algebra system SymPy [15]. To give an example, we now show the computation of one of the terms in Eq. (S64) (other terms similarly).

We start e.g. with the spatial integrals

$$\int d^2x \int d^2x_0 V(\mathbf{x})U(\mathbf{x}_0)j_s^2(\mathbf{x})j_s^2(\mathbf{x}_0)\mathcal{G}(\mathbf{x}, t|\mathbf{x}_0)P_s(\mathbf{x}_0), \quad (\text{S104})$$

where we set $\mathbf{x} = (x_1, x_2)$, $\mathbf{x}_0 = (x_3, x_4)$ such that $j_s^2(\mathbf{x}) = -\Omega x_1$ and $j_s^2(\mathbf{x}_0) = -\Omega x_3$ and we use constants $\{c_i\}$ to write

$$V(\mathbf{x}) = \frac{c_1}{\pi}e^{-c_1((x_1-y_1)^2+(x_2-y_2)^2)} \\ U(\mathbf{x}_0) = \frac{c_2}{\pi}e^{-c_2((x_3-y_3)^2+(x_4-y_4)^2)} \\ P_s(\mathbf{x}_0) = \frac{c_3}{\pi}e^{-c_3(x_3^2+x_4^2)} \\ \mathcal{G}(\mathbf{x}, t|\mathbf{x}_0) = \frac{c_4}{\pi}e^{-c_4((c_5x_3+c_6x_4-x_1)^2+(c_5x_4-c_6x_3-x_2)^2)}. \quad (\text{S105})$$

Integrating from $-\infty$ to ∞ over x_3 and x_4 gives (Gaussian integrals with $c_1, c_2, c_3, c_4 > 0$)

$$\int d^2x_0 V(\mathbf{x})U(\mathbf{x}_0)j_s^2(\mathbf{x})j_s^2(\mathbf{x}_0)\mathcal{G}(\mathbf{x}, t|\mathbf{x}_0)P_s(\mathbf{x}_0) = \frac{\Omega^2 c_1 c_2 c_3 c_4 x_1 (c_2 y_3 + c_4 c_5 x_1 - c_4 c_6 x_2)}{\pi^3 (c_2 + c_3 + c_4 c_5^2 + c_4 c_6^2)^2} \times \\ e^{\frac{c_2^2 y_4^2 + 2c_2 c_4 y_4 (c_5 x_2 y_4 + c_6 x_1) + c_4^2 (c_5 x_2 + c_6 x_1)^2 + (c_2 y_3 + c_4 c_5 x_1 - c_4 c_6 x_2)^2}{c_2 + c_3 + c_4 c_5^2 + c_4 c_6^2} - c_1 x_1^2 + 2c_1 x_1 y_1 - c_1 x_2^2 + 2c_1 x_2 y_2 - c_1 (y_1^2 + y_2^2) - c_2 (y_3^2 + y_4^2) - c_4 (x_1^2 + x_2^2)}. \quad (\text{S106})$$

To integrate over x_1 and x_2 , we simply use ($a_4, a_5 > 0$)

$$\begin{aligned} & \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 (a_1 x_1 + a_2 x_1^2 + a_3 x_1 x_2) e^{-a_4 x_1^2 - a_5 x_2^2 + a_6 x_1 + a_7 x_2 + a_8} \\ &= \frac{\pi (2a_1 a_4 a_5 a_6 + a_2 a_5 (2a_4 + a_6^2) + a_3 a_4 a_6 a_7) e^{\frac{4a_5 a_8 + a_7^2}{4a_5} + \frac{a_6^2}{4a_4}}}{4a_4^{\frac{5}{2}} a_5^{\frac{3}{2}}}. \end{aligned} \quad (\text{S107})$$

Equations for the $\{a_i\}$ in terms of the $\{c_i\}$ can be read off Eq. (S106) and the $\{c_i\}$ contain all parameter dependencies of the process, including the t' . The t' -integration is then performed numerically.

B. Details and simulation parameters for figures in the Letter

The process in Fig. 1 in the Letter is simulated as a free two-dimensional Brownian motion. Numerical Stratonovich integration gives the empirical density and current. The times shown are $\tau_1^- = 1.14$, $\tau_1^+ = 3.83$, $\tau_2^- = 6.54$, $\tau_2^+ = 6.80$.

The process in Fig. 2a-d in the Letter is the shear flow with $\mathbf{F}(x, y) = 2x\hat{\mathbf{y}}$ and $\mathbf{D} = \mathbf{1}$ from $(0, 0)$ to $(2, 0)$ in total time 1. It is simulated with time step size $dt = 0.02$ as Brownian bridge in x -direction (exactly hits 2 after time 1) and then pick trajectories that hit $y_{\text{final}} = 0$ with deviation less than 0.02. Time-reversed and dual reversed trajectories are similarly from $(2, 0)$ to $(0, 0)$ with same or inverted shear. For each transition around than 11,000 – 12,000 trajectories were considered. Arrows are in direction of the first/last step in discretized time.

The trajectory in Fig. 2e in the Letter is sampled from an Ornstein-Uhlenbeck process Eq. (S99) with $\Omega = 3$, $r = D = 1$ and total time $t \approx 37$.

The correlation results shown in Fig. 2f-g in the Letter are computed analytically up to one numerical time integration as described above. The mean trajectory depicted in 2f is, as can be seen from Eq. (S102), given by

$$\mu_{\mathbf{x}_0}(t) = e^{-rt} \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \mathbf{x}_0. \quad (\text{S108})$$

The simulations in Fig. 3 in the Letter are performed with time step $dt = 10^{-4}$ and 8192 repetitions for 3a 4096 repetitions for 3b. All simulations are performed by discretizing Eq. (S99) and sampling the initial point \mathbf{x}_0 from the steady-state distribution $P_s(\mathbf{x})$. Additional parameters in 3c are $h = 1, 0.25, 0.03$ from dark to bright.

-
- [1] C. Maes, K. Netočný, and B. Wynants, Steady state statistics of driven diffusions, *Physica A* **387**, 2675 (2008).
 - [2] E. Mallmin, J. du Buisson, and H. Touchette, Large deviations of currents in diffusions with reflective boundaries (2021), arXiv:2102.04846 [cond-mat.stat-mech].
 - [3] H. Risken, *The Fokker-Planck Equation* (Springer Berlin Heidelberg, 1996).
 - [4] A. Lapolla and A. Godec, Manifestations of projection-induced memory: General theory and the tilted single file, *Front. Phys.* **7**, 182 (2019).
 - [5] H. Qian, A decomposition of irreversible diffusion processes without detailed balance, *J. Math. Phys.* **54**, 053302 (2013).
 - [6] R. Durrett, *Probability: Theory and Examples*, 4th ed. (Cambridge University Press, USA, 2010).
 - [7] C. W. Gardiner, *Handbook of stochastic methods for physics, chemistry, and the natural sciences* (Springer-Verlag, Berlin New York, 1985).
 - [8] A. Lapolla, D. Hartich, and A. Godec, Spectral theory of fluctuations in time-average statistical mechanics of reversible and driven systems, *Phys. Rev. Research* **2**, 043084 (2020).
 - [9] A. Lapolla and A. Godec, Unfolding tagged particle histories in single-file diffusion: exact single- and two-tag local times beyond large deviation theory, *New J. Phys.* **20**, 113021 (2018).
 - [10] M. Kac, On distributions of certain Wiener functionals, *Trans. Am. Math. Soc.* **65**, 1 (1949).
 - [11] W. Feller, *An introduction to probability theory and its applications. Vol. I and II*, Third edition (John Wiley & Sons Inc., New York, 1968).
 - [12] H. Touchette, The large deviation approach to statistical mechanics, *Phys. Rep.* **478**, 1 (2009).
 - [13] A. C. Barato and R. Chetrite, A formal view on level 2.5 large deviations and fluctuation relations, *J. Stat. Phys.* **160**, 1154 (2015).
 - [14] F. Coghi, R. Chetrite, and H. Touchette, Role of current fluctuations in nonreversible samplers (2021), arXiv:2103.14150 [cond-mat.stat-mech].
 - [15] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry, A. R. Terrel, v. Roučka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, and A. Scopatz, SymPy: symbolic computing in Python, *PeerJ Comput. Sci.* **3**, e103 (2017).